WORLD HEALTH ORGANIZATION

LIFE TABLE
AND
MORTALITY ANALYSIS

CHIN LONG CHIANG
# TABLE OF CONTENTS

## FOREWORD

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>XI - XII</td>
<td></td>
</tr>
</tbody>
</table>

## CHAPTER 1. ELEMENTS OF PROBABILITY

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Elements of Probability</td>
<td>1</td>
</tr>
<tr>
<td>2.1. Components</td>
<td>1</td>
</tr>
<tr>
<td>2.2. Definition of Probability</td>
<td>2</td>
</tr>
<tr>
<td>2.3. Examples</td>
<td>2</td>
</tr>
<tr>
<td>2.4. Values of a Probability</td>
<td>3</td>
</tr>
<tr>
<td>2.5. Sure Event and Impossible Event</td>
<td>3</td>
</tr>
<tr>
<td>2.6. Complement of an Event</td>
<td>4</td>
</tr>
<tr>
<td>2.7. Composite Event (A and B)</td>
<td>5</td>
</tr>
<tr>
<td>2.8. Conditional Probability</td>
<td>6</td>
</tr>
<tr>
<td>2.9. Independence</td>
<td>7</td>
</tr>
<tr>
<td>2.10. Multiplication Theorem</td>
<td>8</td>
</tr>
<tr>
<td>2.11. Multiplication Theorem (continuation)</td>
<td>9</td>
</tr>
<tr>
<td>2.12. A Theorem of (Pairwise) Independence</td>
<td>9</td>
</tr>
<tr>
<td>2.13. Composite Event (A or B)</td>
<td>10</td>
</tr>
<tr>
<td>2.14. Mutual Exclusiveness</td>
<td>11</td>
</tr>
<tr>
<td>2.15. Addition Theorem</td>
<td>11</td>
</tr>
<tr>
<td>2.16. Addition Theorem (continuation)</td>
<td>12</td>
</tr>
<tr>
<td>2.17. Summary of the Addition and Multiplication Theorems</td>
<td>12</td>
</tr>
<tr>
<td>2.18. The Distributive Law</td>
<td>12</td>
</tr>
<tr>
<td>2.19. An Example from the Life Table</td>
<td>13</td>
</tr>
<tr>
<td>2.19.1. Conditional Probability</td>
<td>15</td>
</tr>
<tr>
<td>2.19.2. Probabilities of Composite Events</td>
<td>16</td>
</tr>
<tr>
<td>2.19.3. Probability of Dissolution of Marriage</td>
<td>17</td>
</tr>
</tbody>
</table>

## CHAPTER 2. DEATH RATES AND ADJUSTMENT OF RATES

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Age Specific Death Rates</td>
<td>19</td>
</tr>
<tr>
<td>2. Infant Mortality</td>
<td>23</td>
</tr>
<tr>
<td>2.1. Fetal Death Rate (alias &quot;stillbirth rate&quot;)</td>
<td>24</td>
</tr>
<tr>
<td>2.2. Neonatal Mortality Rate</td>
<td>24</td>
</tr>
<tr>
<td>2.3. Perinatal Mortality Rate</td>
<td>24</td>
</tr>
<tr>
<td>2.4. Post Neonatal Mortality Rate</td>
<td>25</td>
</tr>
<tr>
<td>2.5. Infant Mortality Rate</td>
<td>25</td>
</tr>
<tr>
<td>2.6. Fetal Death Ratio</td>
<td>25</td>
</tr>
<tr>
<td>2.7. Maternal Mortality Rate</td>
<td>25</td>
</tr>
<tr>
<td>3. Adjustment of Rates</td>
<td>28</td>
</tr>
<tr>
<td>3.1. Crude Death Rate</td>
<td>30</td>
</tr>
<tr>
<td>3.2. Direct Method Death Rate (D.M.D.R.)</td>
<td>32</td>
</tr>
<tr>
<td>3.3. Comparative Mortality Rate (C.M.R.)</td>
<td>34</td>
</tr>
<tr>
<td>3.4. Indirect Method Death Rate (I.M.D.R.)</td>
<td>35</td>
</tr>
<tr>
<td>3.5. Life Table Death Rate (L.T.D.R.)</td>
<td>36</td>
</tr>
<tr>
<td>3.6. Equivalent Average Death Rate (E.A.D.R.)</td>
<td>38</td>
</tr>
<tr>
<td>3.7. Relative Mortality Index (R.M.I.)</td>
<td>39</td>
</tr>
<tr>
<td>3.8. Mortality Index (M.I.)</td>
<td>39</td>
</tr>
<tr>
<td>3.9. Standardized Mortality Ratio (S.M.R.)</td>
<td>40</td>
</tr>
</tbody>
</table>
CHAPTER 3. STANDARD ERROR OF MORTALITY RATES

1. Introduction
2. The Binomial Distribution
3. Probability of Death and the Age-Specific Death Rate
4. The Death Rate Determined from a Sample
5. Age-Adjusted Death Rates and Mortality Indices
6. Sample Variance of the Age-Adjusted Death Rate
7. Computation of the Sample Variance of the Direct Method Age-Adjusted Death Rate
8. Sample Variance of the Life Table Death Rate

CHAPTER 4. THE LIFE TABLE AND ITS CONSTRUCTION - COMPLETE LIFE TABLES

An Historical Note
1. Introduction
2. Description of the Life Table
3. Construction of the Complete Life Table

CHAPTER 5. THE LIFE TABLE AND ITS CONSTRUCTION - ABRIDGED LIFE TABLES

1. Introduction
2. A Method of Life Table Construction
3. The Fraction of the Last Year of Life, \( a'_x \), and the Fraction of the Last Age Interval of Life, \( a_1 \)
   3.1. The Fraction of the Last Year of Life, \( a'_x \)
   3.2. The Fraction of the last Age Interval of Life, \( a_1 \)
4. Significant Historical Contributions to the Construction of Abridged Life Tables
   4.1. King's Method
   4.2. Reed-Merrell Method
   4.3. Greville's Method
   4.4. Wiesler's Method
   4.5. Sirken's Method
   4.6. Keyfitz's Method
5. Cohort (Generation) Life Table
CHAPTER 6. STATISTICAL INFERENCE REGARDING LIFE TABLE FUNCTIONS

1. Introduction
2. The Probability of Dying \( q_i \) and the Probability of Surviving \( p_i \)
3. The Survival Probability, \( p_{ij} \)
4. Expectation of life at age \( x_a, e_a \)
   4.1. Formula for the Variance of the Expectation of Life
   4.2. Computation of the Variance of the Expectation of Life in a Current Life Table
   4.3. Statistical Inference About Expectation of Life

CHAPTER 7. MULTIPLE DECREMENT TABLE FOR A CURRENT POPULATION

1. Introduction
   1.1. Crude Probability
   1.2. Net Probability
   1.3. Partial Crude Probability
2. Computation of the Crude Probability, \( Q_i \)
   2.1. Information Needed from a Current Population
   2.2. Computation of Rates and Probabilities
   2.3. Computation of Standard Deviation
3. Multiple Decrement Tables for Sweden and Australia Populations
4. Interpretation of a Multiple Decrement Table

CHAPTER 8. THE LIFE TABLE WHEN A PARTICULAR CAUSE IS ELIMINATED

1. Introduction
2. Computation of the Net Probability, \( \hat{q}_{i,1} \)
3. Construction of the Life Table
4. Interpretation of Findings
   4.1. Comparison of Impact on Human Mortality of Three Major Causes of Death: All Accidents, Cancer All Forms and Cardiovascular-Renal Diseases
   4.2. Cancer All Forms
5. The Life Table When a Particular Cause Alone is Operating in a Population
CHAPTER 9. MEDICAL FOLLOW-UP STUDIES

1. Introduction 193
2. Estimation of Probability of Survival and Expectation of Life 195
   2.1. Basic Random Variables and Likelihood Functions 195
   2.2. Maximum-Likelihood Estimators of the Probabilities \( p_x \) and \( q_x \) 200
   2.3. Estimation of Survival Probability 201
   2.4. Estimation of the Expectation of Life 202
   2.5. Sample Variance of the Observed Expectation of Life 204
   2.6. An Example of Life Table Construction for a Follow-up Population 205
3. Consideration of Competing Risks 213
   3.1. Basic Random Variables and Likelihood Functions 215
   3.2. Estimation of Crude, Net, and Partial Crude Probabilities 218
   3.3. An Example 220
4. Lost Cases 223

APPENDICES

APPENDIX I. THEORETICAL JUSTIFICATION OF THE METHOD OF LIFE TABLE CONSTRUCTION IN CHAPTER 3 227

APPENDIX II. STATISTICAL THEORY OF LIFE TABLE FUNCTIONS 231

1. Introduction 231
2. Probability Distribution of \( L_x \), the Number of Survivors at Age \( x \) 234
   2.1. Mortality Laws 237
      (i) Gompertz distribution 237
      (ii) Makeham's distribution 238
      (iii) Weibull's distribution 238
      (iv) Exponential distribution 239
3. Joint Probability Distribution of the Number of Survivors 240
4. Joint Probability Distribution of the Number of Deaths 242
5. Optimum Properties of \( \hat{p}_j \) and \( \hat{q}_j \) 244
   5.1. Maximum Likelihood Estimator of \( p_j \) 244
6. Distribution of \( \hat{e}_{x} \), the Observed Expectation of Life at Age \( x \) 247
   6.1. The Variance of the Expectation of life, \( \hat{e}_{x} \) 250
<table>
<thead>
<tr>
<th>APPENDIX III - THE THEORY OF COMPETING RISKS</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>An Historical Note - Daniel Bernoulli's Work</td>
<td>255</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>255</td>
</tr>
<tr>
<td>2. Relation Between Crude, Net and Partial Crude Probabilities</td>
<td>261</td>
</tr>
<tr>
<td>2.1. Relation Between Crude and Net Probabilities</td>
<td>264</td>
</tr>
<tr>
<td>2.2. Relation Between Crude and Partial Crude Probabilities</td>
<td>268</td>
</tr>
<tr>
<td>3. Competing Risks with Interaction</td>
<td>270</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX IV - MULTIPLE DECREMENT TABLES</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>277</td>
</tr>
<tr>
<td>2. The Chain Multinomial Distributions</td>
<td>281</td>
</tr>
<tr>
<td>3. Estimation of the Crude Probabilities</td>
<td>285</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX V - Fraction of Last Age Interval of Life $a_i$</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>289</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX VI-A - COMPUTER PROGRAM FOR ABRIDGED LIFE TABLE CONSTRUCTION</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>315</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>APPENDIX VI-B - COMPUTER PROGRAM FOR LIFE TABLE CONSTRUCTION WHEN A PARTICULAR CAUSE OF DEATH IS ELIMINATED</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>323</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>REFERENCES</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>332</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GLOSSARY</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>349</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 1</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>351</td>
</tr>
<tr>
<td>CHAPTER 2</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 3</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 4</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 5</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 6</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 7</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 8</td>
<td></td>
</tr>
<tr>
<td>CHAPTER 9</td>
<td></td>
</tr>
<tr>
<td>APPENDIX I</td>
<td></td>
</tr>
<tr>
<td>APPENDIX II</td>
<td></td>
</tr>
<tr>
<td>APPENDIX III</td>
<td></td>
</tr>
<tr>
<td>APPENDIX IV</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF TABLES

CHAPTER 1 - ELEMENTS OF PROBABILITY

Table 1. The number of survivors and the number died out of 100,000 live births

CHAPTER 2 - DEATH RATES AND ADJUSTMENT OF RATES

Table 1. Fetal death and infant mortality
Table 2. Infant mortality rates and percent reduction: Selected countries, 1935, 1950, and 1962
Table 3. Fetal mortality rates and percent reduction, selected countries, 1955 and 1963
Table 4. Fetal and infant mortality and percent reduction, United States, 1960 and 1970
Table 5. Age-adjusted rates and mortality indices
Table 6. Age-specific death rates and crude death rates for communities A and B
Table 7. Direct method age-adjusted rates for communities A and B

CHAPTER 3 - STANDARD ERROR OF MORTALITY RATES

Table 1. Formulas and weights used to compute the crude death rate, age-adjusted rates and mortality indices
Table 2. Computation of sample standard error of the age-adjusted death rate for total population, California, 1970.

CHAPTER 4 - THE LIFE TABLE AND ITS CONSTRUCTION - COMPLETE LIFE TABLES

Table 1. Construction of complete life table for total California population, USA, 1970
Table 2. Complete life table for total California population, USA, 1970

CHAPTER 5 - THE LIFE TABLE AND ITS CONSTRUCTION - ABRIDGED LIFE TABLES

Table 1. Construction of abridged life table for total California population, USA, 1970
Table 2. Abridged life table for total California population, USA, 1970
Table 3. Frequency distribution of deaths by interval of days lived in the last year of life of selected ages, total population California, 1960
CHAPTER 5 (continued)

Table 4. Computation of the fraction $a_0$ based on infant deaths, United States total population, 1970

Table 5. Computation of the fraction $a_1$ for age interval (1,5) based on California mortality data, 1970

Table 6. Fraction of last age interval of life, $a_1^1$, for selected populations

Table 7. Computation of $a_1$ for age interval (5,10) and (10,15) based on California population, 1970

Table 8. Life table of adult male Drosophila Melanogaster

Table 9. Life table of adult female Drosophila Melanogaster

CHAPTER 6 - STATISTICAL INFERENCE REGARDING LIFE TABLE FUNCTIONS

Table 1. Estimate of probability of dying in the first year of life and the standard error, California, 1960 and 1970

Table 2. Abridged life table for total United States population, 1960

Table 3. Computation of the standard error or survival probability. Total United States population, 1960

Table 4. Computation of standard error of survival probability. Total California population, 1970

Table 5. Statistical test for the significance of difference between survival probabilities of United States population, 1960, and California population, 1970

Table 6. Computation of the sample variance of the observed expectation of life. Total U.S. population, 1960

Table 7. Expectation of life and the standard error, total United States population, 1960, and total California population, 1970

CHAPTER 7 - MULTIPLE DECREMENT TABLE FOR A CURRENT POPULATION

Table 1. Multiple decrement table - The crude probability of dying ($\hat{q}_{i+}$) from a specific cause ($R_6$) in age interval ($x_i,x_{i+1}$)

Table 2. Computation of the crude probability of dying from a specific cause and the corresponding standard error. Sweden population, age interval (1,5), 1967
CHAPTER 7 (continued)

Table 3. Mid-year population and deaths by age and cause - Sweden, 1967  
Table 4. Mid-year population and deaths by age and cause - Australia, 1967  
Table 5. Multiple decrement table for selected causes of death and the standard error of the crude probability of dying - Sweden population, 1967  
Table 6. Multiple decrement table for selected causes of death and the standard error of the crude probability of dying - Australia population, 1967

CHAPTER 8 - THE LIFE TABLE WHEN A PARTICULAR CAUSE IS ELIMINATED

Table 1. Computation of the net probability of dying, $q_{i,l'}$ when cardiovascular-renal (CVR) diseases ($R_1$) are eliminated as a cause of death, white males, United States, 1960  
Table 2. Abridged life table when cardiovascular renal diseases are eliminated as a cause of death for white males, United States, 1960  
Table 3. Abridged life table when cardiovascular renal diseases are eliminated as a cause of death for white females, United States, 1960  
Table 5. Probability of survival and the effect of eliminating CVR diseases as a cause of death. White males and females, U.S., 1960  
Table 7. Life table of the Federal Republic of Germany population, 1970, when cardiovascular diseases ($R_1$) are eliminated as a cause of death  
Table 8. Life table of the Federal Republic of Germany population, 1970, when cancer all forms ($R_2$) is eliminated as a cause of death.  
Table 9. Life table of the Federal Republic of Germany population, 1970, when all accidents ($R_3$) are eliminated as a cause of death
CHAPTER 8 

Table 10a. Probability of dying when cardiovascular diseases ($R_1$), cancer all forms ($R_2$), or all accidents ($R_3$) is eliminated as a cause of death - The Federal Republic of Germany, 1970

Table 10b. Probability of dying and the effect of eliminating cardiovascular diseases ($R_1$), cancer all forms ($R_2$), or all accidents ($R_3$) as a cause of death in each age interval - The Federal Republic of Germany, 1970

Table 11. The expectation of life and the effect of elimination of cardiovascular diseases ($R_1$), cancer all forms ($R_2$) or all accidents as a cause of death in each age interval. The Federal Republic of Germany, 1970

Table 12. Probability of dying and the effect of eliminating cancer all forms ($R_2$) as a cause of death - (Canada 1968 and France 1969)

Table 13. Expectation of life and the effect of eliminating cancer all forms ($R_2$) as a cause of death - Canada 1968 and France 1969

CHAPTER 9 - MEDICAL FOLLOW-UP STUDIES

Table 1. Distribution of $N_x$ patients according to withdrawal status and survival status in the interval $(x, x+1)$

Table 2. Comparison between $p_x^{1/5}$ and $-(1-p_x)/\ln p_x$

Table 3. Survival experience following diagnosis of cancer of the cervix uteri - Cases initially diagnosed 1942-1954, California, U.S.A.

Table 4. Life table of patients diagnosed as having cancer of the cervix uteri - Cases initially diagnosed 1942-1954, California, U.S.A.

Table 5. Survival experience after diagnosis of cancer of the cervix uteri - Cases initially diagnosed 1942-1954, California, U.S.A. - The main life table functions and their standard errors

Table 6. Distribution of $N_x$ patients according to withdrawal status, survival status, and cause of death in the interval $(x, x+1)$
CHAPTER 9 (continued)

Table 7. Survival experience following diagnosis of cancer of the cervix uteri - Cases initially diagnosed 1942-1954, California, U.S.A. 221

Table 8. Survival experience after diagnosis of cancer of the cervix uteri - Cases initially diagnosed 1942-1954, California, U.S.A. - Estimated crude and net probabilities of death from cancer of the cervix uteri and from other causes 222

APPENDIX II - STATISTICAL THEORY OF LIFE TABLE FUNCTIONS

Table 1. Life table 232

APPENDIX III - THE THEORY OF COMPETING RISKS

Table 1. Crude probabilities and intensity functions 264

APPENDIX IV - MULTIPLE DECREMENT TABLES

Table 1. Multiple Decremental Table 288

APPENDIX V - FRACTION OF LAST AGE INTERVAL OF LIFE, $a_i$

Table 1. Austria, 1969 289
Table 2. California, 1960 290
Table 3. Canada, 1968 291
Table 4. Costa Rica, 1963 292
Table 5. Finland, 1968 293
Table 6. France, 1969 294
Table 7. East Germany, 1967 295
Table 8. West Germany, 1969 296
Table 9. Hungary, 1967 297
Table 10. Ireland, 1966 298
Table 11. North Ireland, 1966 299
Table 12. Italy, 1966 300
Table 13. The Netherlands, 1968 301
Table 14. Norway, 1968 302
Table 15. Okinawa, 1960 303
APPENDIX V (continued)

Table 16. Panama, 1968 304
Table 17. Portugal, 1960 305
Table 18. Romania, 1965 306
Table 19. Scotland, 1968 307
Table 20. Spain, 1965 308
Table 21. Sri Lanka, 1952 309
Table 22. Sweden, 1966 310
Table 23. Switzerland, 1968 311
Table 24. United States of America, 1970 312
Table 25. Yugoslavia, 1968 313
LIFE TABLE AND MORTALITY ANALYSIS

FOREWORD

This publication on advanced methods of analysis of the mortality of populations is the second in a series of teaching aids addressed to a wide spectrum of professionals in the health field, statistics and demography. The first, a Manual on methods of analysis of national mortality statistics for public health purposes, was published in 1977 and focused on basic methods of analysis which are commonly used in National Statistical Departments.

Public Health and Demography owe so much to the quantitative study of mortality. For centuries, the primary determinant of population trends has been mortality and it still remains so in many less developed countries; it was mortality that formed the primary challenge to the medical professions; it was the prevention of early death that was the primary objective of public health workers and of social legislation. Nowadays, this central role of the study of mortality has gradually yielded way to concern for other phenomena such as fertility and morbidity and the definition of positive health and the study of the provision and use of health services. Nonetheless, the analysis of mortality data is still an indispensable part of informed decision-making and of the evaluation of policies on health services. New problems have arisen even in the area of mortality analysis; the growing importance of chronic diseases has raised new issues and problems; demands for statistical analysis have become ever more sophisticated; the improved quality of certification of the causes of death has created a demand for a detailed study of the difficulties encountered in their interpretation; the use of computers has changed the problems of data processing and facilitated more complex methods of analysis. It was with these considerations in mind that the work on an up-to-date publication on mortality analysis was initiated.

This volume emphasizes the more advanced methods in the study of survival and mortality. The life table method of analysis, historically rooted in the actuarial and demographic sciences has by now become an indispensable tool for investigators in other disciplines such as epidemiology, zoology, manufacturing etc. The classical concept of counting risks is introduced and integrated into a coherent probabilistic approach to the study of a broad range of processes with a stochastic distribution of exit from one or more competing
causes with the life table as central theme. Follow-up studies with due attention to truncated information are of great practical importance not only for medical research but may prove particularly useful for health statisticians in less developed countries who - in the absence of complete nation-wide vital statistics - concentrate on the study of the survival experience of relatively small population groups.

It is hoped that this volume will be of use for post-graduate courses in biometry, demography and epidemiology, and together with the manual will also serve as a background for training activities and refresher courses in health statistics organized or sponsored by the World Health Organization. In fact, part of the manuscript has been tested in courses organized by the World Health Organization with the financial support of UNFPA; the experience gained in this practical application is reflected in the text.

This volume has been prepared by Professor Chin Long Chiang, University of California, Berkeley (U.S.A.) an outstanding authority and pioneer in the application of the stochastic approach to the study of death processes. The manuscript has also profited from the comments of the United Nations' Population Division, Professors H. Campbell (U.K.) and S. Koller (Federal Republic of Germany) and various staff members of the World Health Organization such as the statistical officers in the Regional Offices. Dr H. Hansluwka, World Health Organization was most actively involved in the design of this volume and coordinated the various activities which led to the production of this volume.

ACKNOWLEDGEMENTS

The Support of the United Nations Fund for Population Activities is gratefully acknowledged.

This manual was made possible with the help of the following persons: Flora Fung, Bonnie Hutchings, Linda Kwok, Carol Langhauser, Patrick Wong, and Rodney Wong. Their assistance is very much appreciated.
CHAPTER 1
ELEMENTS OF PROBABILITY

1. Introduction

A good understanding of the basic concept of probability is essential for proper analysis of mortality data. Because of its potential as an analytic tool, probability has been increasingly used in vital statistics and life table analyses. As a result, studies of vital data are no longer limited to a mere description or interpretation of numerical values; statistical inference can be made regarding mortality and survival patterns of an entire population. While it is a mathematical concept, probability has an interesting intuitive appeal. Many natural phenomena can be described by means of probability laws; occurrence of daily events also seems to follow a definite pattern. Even such spontaneous events as accidents can be predicted in advance with a certain degree of accuracy. Mortality laws proposed by Benjamin Gompertz in 1825 and by W. M. Makeham in 1860 have been used in studies of human survival and death both in the field of health and in the actuarial sciences. It is appropriate then to begin this manual by introducing the fundamental probability concept, related formulas and illustrative examples.

2. Elements of Probability

2.1. Components. The concept of probability involves three components: (a) a random experiment, (b) possible outcomes, and (c) an event of interest. A random experiment is an experiment that has a number of possible outcomes, but it is not certain which of the outcomes will occur before the experiment is performed. Thus, in speaking of probability, one must have in mind a random experiment under consideration and an event of interest.
2.2. Definition of probability. The probability of the occurrence of event A is defined as the ratio of the number of outcomes where event A occurs to the total number of outcomes. For simplicity, we shall use the term "the probability of event A" for "the probability of the occurrence of event A."

Suppose that a random experiment may result in a number \( n \) of possible (and equally likely) outcomes, and in \( n(A) \) of these outcomes event A occurs. Then the probability of event A is defined as follows:

\[
\Pr\{A\} = \frac{n(A)}{n}.
\]  
(2.1)

Thus, the probability of event A in a random experiment is a measure of the likelihood of occurrence of the event.

2.3. Examples. The following examples may elucidate the concept of probability.

Example 1. In tossing a fair coin once, what is the probability of a head turning up? Here, tossing a fair coin once is the random experiment, and the possible outcomes are a head and a tail. Let event A be "a head." The number of possible outcomes, \( n \), is 2, and the number of outcomes where a head occurs, \( n(A) \), is 1. Therefore, the probability is

\[
\Pr\{A\} = \frac{n(A)}{n} = \frac{1}{2}.
\]

Example 2. In rolling a fair die once, there are 6 possible outcomes. Let event A be 3 dots. Here \( n=6 \) and \( n(A) = 1 \); therefore:

\[
\Pr\{A\} = \frac{n(A)}{n} = \frac{1}{6}.
\]

Let event B be an even number of dots, with \( n(B) = 3 \). The probability of B is:
Example 3. A name is drawn at random from a group of 120 people consisting of 39 females and 81 males. Let event A be the drawing of a female name. The probability of event A is:

\[ P(A) = \frac{n(A)}{n} = \frac{39}{120} = \frac{13}{40} \]

Example 4. A list of n = 100 names consists of n(s) = 98 names of survivors and n(d) = 2 of those who have died. A name is drawn at random from the list. The probability that the name drawn will be that of a survivor is

\[ P(s) = \frac{n(s)}{n} = \frac{98}{100} = 0.98 \]

and that of one who has died is

\[ P(d) = \frac{n(d)}{n} = \frac{2}{100} = 0.02 \]

Clearly, the sum of the two probabilities is unity:

\[ P(s) + P(d) = 0.98 + 0.02 = 1 \]

2.4. Values of a probability. From the definition we see that the probability of an event A is an (idealized) proportion or relative frequency. Thus, a probability can only take on values between zero and one, i.e.,

\[ 0 < Pr(A) < 1 \] (2.2)

2.5. Sure event and impossible event. A sure event is an event that always occurs. If I is a sure event, then

\[ Pr(I) = 1 \] (2.3)
An impossible event is an event that never occurs. If $\emptyset$ is an impossible event, then

$$\Pr(\emptyset) = 0 .$$  \hspace{1cm} (2.4)

2.6. **Complement of an event** (or negation of an event) can be best illustrated with examples. Let $\bar{A}$ be the complement of event $A$.

<table>
<thead>
<tr>
<th>Example</th>
<th>$A$</th>
<th>$\bar{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sex of a baby</td>
<td>male</td>
<td>female</td>
</tr>
<tr>
<td>Toss of a coin</td>
<td>head</td>
<td>tail</td>
</tr>
<tr>
<td>Toss of a die</td>
<td>3 dots</td>
<td>anything but 3 dots</td>
</tr>
<tr>
<td>Toss of a die</td>
<td>even no. of dots</td>
<td>odd number of dots</td>
</tr>
<tr>
<td>Survival analysis</td>
<td>survival</td>
<td>death</td>
</tr>
</tbody>
</table>

Thus, the complement $\bar{A}$ occurs when and only when event $A$ does not occur. In a random experiment the total number of outcomes can be divided into two groups according to the occurrence of $A$ or of $\bar{A}$,

$$n = n(A) + n(\bar{A}) .$$

The probability of $\bar{A}$ in a random experiment is, by definition,

$$\Pr(\bar{A}) = \frac{n(\bar{A})}{n} .$$

It is clear then that, whatever event $A$ may be,

$$\Pr(A) + \Pr(\bar{A}) = 1$$  \hspace{1cm} (2.5)

or
Pr(A) = 1 - Pr(A)  \hspace{1cm} (2.5a)

In words, the probability of the complement of A is equal to the complement of the probability of A.

2.7. Composite event (A and B). Given two events A and B, we define a composite event \( A \cap B \) (or \( AB \) for simplicity) by saying that the event \( AB \) occurs if both event A and event B occur.

Example 5. Consider a group of 200 newborn babies divided according to sex and prematurity as shown in the following 2x2 table:

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
<th>Marginal row total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Premature</td>
<td>11</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>( B )</td>
<td>( n(AB) )</td>
<td>( n(AB) )</td>
<td>( n(B) )</td>
</tr>
<tr>
<td>Full term</td>
<td>93</td>
<td>87</td>
<td>180</td>
</tr>
<tr>
<td>( \bar{B} )</td>
<td>( n(AB) )</td>
<td>( n(AB) )</td>
<td>( n(AB) )</td>
</tr>
<tr>
<td>Marginal</td>
<td>104</td>
<td>96</td>
<td>200</td>
</tr>
<tr>
<td>Column Total</td>
<td>( n(A) )</td>
<td>( n(\bar{A}) )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

Let \( A = \text{male}, \quad \bar{A} = \text{female}, \quad B = \text{premature}, \quad \bar{B} = \text{full term}. \)

A baby is picked at random from the group; the composite event \( AB \) is a premature boy. The corresponding probability is

\[
Pr(AB) = \frac{n(AB)}{n} = \frac{11}{200} \hspace{1cm} (2.6)
\]

Other possible composite events are

\( \bar{A}B = \text{a full term boy} \)
\( \bar{A}B = \text{a premature girl} \)
\( AB = \text{a full term girl} \)
The probabilities \( \Pr\{AB\} \), \( \Pr\{\overline{A}B\} \), and \( \Pr\{A\overline{B}\} \) can be computed from the above table.

If \( I \) is a sure event, then

\[
\Pr\{AI\} = \Pr\{A\} \quad .
\]  

If \( \emptyset \) is an impossible event, then

\[
\Pr\{A\emptyset\} = 0 \quad .
\]

2.8. Conditional probability. The conditional probability of \( B \) given that \( A \) has occurred is defined by:

\[
\Pr\{B|A\} = \frac{\Pr\{AB\}}{\Pr\{A\}} \quad .
\]  

Since

\[
\Pr\{AB\} = \frac{n(AB)}{n} \quad \text{and} \quad \Pr\{A\} = \frac{n(A)}{n} \quad ,
\]

we have

\[
\Pr\{B|A\} = \frac{n(AB)/n}{n(A)/n} = \frac{n(AB)}{n(A)} \quad .
\]

In terms of the previous example, \( \Pr\{B|A\} \) is the probability that a baby chosen at random from the boys will be premature. Since there are \( n(A) = 104 \) boys, and among them \( n(AB) = 11 \) are premature, we have

\[
\Pr\{B|A\} = \frac{n(AB)}{n(A)} = \frac{11}{104} \quad ,
\]

or, using

\[
\Pr\{AB\} = \frac{n(AB)}{n} = \frac{11}{200} \quad \text{and} \quad \Pr\{A\} = \frac{n(A)}{n} = \frac{104}{200} \quad ,
\]

and
we obtain the same value.

It is clear that the conditional probability $\Pr(B|A)$ is different from the conditional probability $\Pr(A|B)$. In the above example the probability that a premature baby will be a boy is computed from

$$\Pr(A|B) = \frac{n(AB)}{n(B)} = \frac{11}{20} .$$

The reader is advised to use the above example to compute and interpret the following conditional probabilities: $\Pr(B|\bar{A})$, $\Pr(\bar{B}|A)$, $\Pr(\bar{B}|\bar{A})$, $\Pr(A|B)$, $\Pr(A|\bar{B})$, and $\Pr(\bar{A}|\bar{B})$.

In applying conditional probability to a practical problem, one should beware of a sequence that may exist in the occurrence of events. If event $A$ occurs before event $B$, then the conditional probability $\Pr(A|B)$ may not be meaningful, whereas the conditional probability $\Pr(B|A)$ is meaningful. For example, in a study of sex differential infant mortality, sex of infant, male ($A$) or female ($\bar{A}$), is determined before mortality in the first year of life (denoted by $B$) occurs. Comparison of infant mortality of males with that of females requires the conditional probabilities $\Pr(B|A)$ and $\Pr(B|\bar{A})$. But it may be difficult to comprehend the conditional probability $\Pr(A|B)$ that an infant who dies will be male.

2.9. **Independence.** Event $B$ is said to be independent of event $A$ if the conditional probability of $B$ given $A$ is equal to the (absolute) probability of $B$. In formula

$$\Pr(B|A) = \Pr(B) .$$

This means that the likelihood of the occurrence of $B$ is not influenced by the occurrence of $A$. Clearly, if $B$ is independent of $A$, $B$ is also
independent of $A$, or

$$\Pr(B|A) = \Pr(B) = \Pr(B|\overline{A}) \quad (2.12)$$

Let $A = \text{male}$, $B = \text{prematurity}$. If

$$\Pr\{\text{premature baby}\mid \text{male}\} = \Pr\{\text{premature baby}\},$$

then

$$\Pr\{\text{premature baby}\mid \text{female}\} = \Pr\{\text{premature baby}\},$$

and we say that prematurity is independent of sex of the baby.

To verify whether an event $B$ is independent of an event $A$ in a particular problem, we compute separately

$$\Pr(B\mid A) \quad \text{and} \quad \Pr(B).$$

If the two numerical values are equal, we say that $B$ is independent of $A$.

In the example in section 2.7

$$\Pr(B\mid A) = \frac{11}{104} \quad \text{and} \quad \Pr(B) = \frac{20}{200}.$$ 

Since $11/104$ is not equal to $20/200$, according to the information given in this example, prematurity is dependent on the sex of a baby.

2.10. Multiplication theorem. The probability of $AB$ is equal to the product of probability of $A$ and the conditional probability of $B$ given $A$, or

$$\Pr(AB) = \Pr(A) \times \Pr(B\mid A) \quad (2.13)$$

Proof:

$$\Pr(AB) = \frac{n(AB)}{n} = \frac{n(A)}{n} \times \frac{n(AB)}{n(A)} = \Pr(A) \times \Pr(B\mid A).$$
With reference to the 2x2 table in example 5, we see that

\[ \Pr(AB) = \frac{11}{200} \]

and

\[ \Pr(A) \times \Pr(B|A) = \frac{104}{200} \times \frac{11}{104} = \frac{11}{200} \]

therefore

\[ \Pr(AB) = \Pr(A) \times \Pr(B|A) \]

Since event AB is the same as event BA, the multiplication theorem has an alternative formula:

\[ \Pr(AB) = \Pr(B) \times \Pr(A|B) \quad (2.14) \]

The formulas of the multiplication theorem for three and four events are

\[ \Pr(ABC) = \Pr(A) \times \Pr(B|A) \times \Pr(C|AB) \quad (2.15) \]

and

\[ \Pr(ABCD) = \Pr(A) \times \Pr(B|A) \times \Pr(C|AB) \times \Pr(D|ABC) \quad (2.16) \]

2.11. Multiplication theorem (continuation). If events are independent, then the formulas of the multiplication theorem become

\[ \Pr(AB) = \Pr(A) \times \Pr(B) \quad (2.17) \]
\[ \Pr(ABC) = \Pr(A) \times \Pr(B) \times \Pr(C) \quad (2.18) \]
\[ \Pr(ABCD) = \Pr(A) \times \Pr(B) \times \Pr(C) \times \Pr(D) \quad (2.19) \]

2.12. A theorem of (pairwise) independence. If B is independent of A, then A is independent of B, and A and B are said to be independent events. Symbolically, the theorem may be stated as follows:
If \( \Pr\{B|A\} = \Pr\{B\} \)
then
\[ \Pr\{A|B\} = \Pr\{A\} . \]

**Proof:** According to the multiplication theorem,

\[ \Pr\{AB\} = \Pr\{A\} \times \Pr\{B|A\} \text{ and } \Pr\{AB\} = \Pr\{B\} \times \Pr\{A|B\} . \]

It follows that

\[ \Pr\{A\} \times \Pr\{B|A\} = \Pr\{B\} \times \Pr\{A|B\} . \]

(2.20)

If \( B \) is independent of \( A \) so that \( \Pr\{B|A\} = \Pr\{B\} \), then (2.20) becomes

\[ \Pr\{A\} \times \Pr\{B\} = \Pr\{B\} \times \Pr\{A|B\} , \]

and consequently

\[ \Pr\{A\} = \Pr\{A|B\} . \]

Conversely, if \( B \) is dependent of \( A \), then \( A \) is dependent of \( B \).

In the example in part 7,

\[ \Pr\{B|A\} = \frac{11}{104} \text{ and } \Pr\{B\} = \frac{20}{200} \]

so that \( B \) is dependent of \( A \), while

\[ \Pr\{A|B\} = \frac{11}{20} \text{ and } \Pr\{A\} = \frac{104}{200} , \]

so that \( A \) is dependent of \( B \).

2.13. Composite event (\( A \) or \( B \)). By a composite event \( A \) or \( B \) we mean either \( A \) or \( B \) or both. Thus the event \( A \) or \( B \) occurs if either \( A \) occurs, or \( B \) occurs, or \( AB \) occurs.
2.14. **Mutual exclusiveness.** Two events are said to be mutually exclusive if the occurrence of one implies the non-occurrence of the other; in other words, they cannot occur simultaneously in a single experiment. If A and B are mutually exclusive events, then \( n(AB) = 0 \) and \( \Pr{AB} = 0 \).

2.15. **Addition theorem.**

\[
\Pr{A \text{ or } B} = \Pr{A} + \Pr{B} - \Pr{AB}
\]

**Proof:** Using the example in part 2.7 again and by direct enumeration, we see that

\[
\Pr{A \text{ or } B} = \frac{n(A) + n(B) - n(AB)}{n}
\]

Dividing every term in the numerator by the denominator, we have

\[
\Pr{A \text{ or } B} = \frac{n(A)}{n} + \frac{n(B)}{n} - \frac{n(AB)}{n} = \Pr{A} + \Pr{B} - \Pr{AB}
\]

(2.21)

**Example:** Let \( A = \text{male} \), \( B = \text{prematurity} \). From example 5 in section 2.7, we compute

\[
\Pr{A \text{ or } B} = \Pr{A} + \Pr{B} - \Pr{B}
\]

\[
= \frac{104}{200} + \frac{20}{200} - \frac{11}{200} = \frac{113}{200}
\]

The formulas of the addition theorem for three and four events are

\[
\Pr{A \text{ or } B \text{ or } C} = \Pr{A} + \Pr{B} + \Pr{C} - \Pr{AB} - \Pr{BC} - \Pr{CA} + \Pr{ABC}
\]

(2.22)

\[
\Pr{A \text{ or } B \text{ or } C \text{ or } D} = \Pr{A} + \Pr{B} + \Pr{C} + \Pr{D} - \Pr{AB} - \Pr{AC} - \Pr{AD} - \Pr{BC} + \Pr{BD} - \Pr{CD} + \Pr{ABC} + \Pr{ABD} + \Pr{ACD} + \Pr{BCD} - \Pr{ABCD}
\]

(2.23)
2.16. Addition theorem (continuation). When events are mutually exclusive so that \( \Pr\{AB=0\} \), etc., then the formulas of the addition theorem become

\[
\Pr\{A \text{ or } B\} = \Pr\{A\} + \Pr\{B\} \\
\Pr\{A \text{ or } B \text{ or } C\} = \Pr\{A\} + \Pr\{B\} + \Pr\{C\} \\
\Pr\{A \text{ or } B \text{ or } C \text{ or } D\} = \Pr\{A\} + \Pr\{B\} + \Pr\{C\} + \Pr\{D\}
\]

and so on.

2.17. Summary of the addition and multiplication theorems. Simple as they may appear to be, the addition and multiplication theorems are indispensible in computing probabilities. The following table is prepared to facilitate the applications of these two theorems.

<table>
<thead>
<tr>
<th>Which theorem</th>
<th>Multiplication theorem</th>
<th>Addition theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>When to use</td>
<td>A and B</td>
<td>A or B</td>
</tr>
<tr>
<td>Theorem</td>
<td>( \Pr{AB} = \Pr{A} \times \Pr{B} )</td>
<td>( \Pr{A \text{ or } B} = \Pr{A} + \Pr{B} - \Pr{AB} )</td>
</tr>
<tr>
<td>Are the events</td>
<td>independent?</td>
<td>mutually exclusive?</td>
</tr>
<tr>
<td>Particular form of theorem</td>
<td>If independent, then ( \Pr{AB} = \Pr{A} \times \Pr{B} )</td>
<td>If mutually exclusive, then ( \Pr{A \text{ or } B} = \Pr{A} + \Pr{B} )</td>
</tr>
</tbody>
</table>

2.18. The distributive law. When the computation of a probability requires both the addition and multiplication theorems, the rule of application of the two theorems is similar to that in an arithmetic problem. The most useful rule of operation is the distributive law:

\[
2(3 + 4) = 2 \times 3 + 2 \times 4
\]

in an arithmetic problem, and

\[
\Pr\{A(B \text{ or } C)\} = \Pr\{AB \text{ or } AC\}
\]

in probability; or
(2 + 3)(4 + 5) = 2 \times 4 + 2 \times 5 + 3 \times 4 + 3 \times 5

and

\[ \Pr\{(A \text{ or } B)(C \text{ or } D)\} = \Pr\{AC \text{ or } AD \text{ or } BC \text{ or } BD\} \]  \hspace{1cm} (2.28)

Using example 5 once again, we have

\[ \Pr\{A(B \text{ or } \overline{B})\} = \Pr\{AB \text{ or } A\overline{B}\} = \Pr\{AB\} + \Pr\{A\overline{B}\} = \frac{11}{200} + \frac{93}{200} = 0.054 \]

In this case, \((B \text{ or } \overline{B}) = \Omega\) is a sure event,

\[ \Pr\{A(B \text{ or } \overline{B})\} = \Pr\{A\Omega\} = \Pr\{A\} = \frac{104}{200} \]

2.19. An example from the life table

Table 1. The number of survivors and the number died out of 100,000 live births

<table>
<thead>
<tr>
<th>Age Interval (in years) (x_i) to (x_{i+1})</th>
<th>Number living at age (x_i) ((l_i))</th>
<th>Number dying in interval ((d_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>1000000</td>
<td>1801</td>
</tr>
<tr>
<td>1-5</td>
<td>98199</td>
<td>316</td>
</tr>
<tr>
<td>5-10</td>
<td>97883</td>
<td>184</td>
</tr>
<tr>
<td>10-15</td>
<td>97699</td>
<td>183</td>
</tr>
<tr>
<td>15-20</td>
<td>97516</td>
<td>550</td>
</tr>
<tr>
<td>20-25</td>
<td>96966</td>
<td>750</td>
</tr>
<tr>
<td>25-30</td>
<td>96216</td>
<td>681</td>
</tr>
<tr>
<td>30-35</td>
<td>95535</td>
<td>766</td>
</tr>
<tr>
<td>35-40</td>
<td>94769</td>
<td>1060</td>
</tr>
<tr>
<td>40-45</td>
<td>93709</td>
<td>133</td>
</tr>
<tr>
<td>45-50</td>
<td>92126</td>
<td>24.</td>
</tr>
<tr>
<td>50-55</td>
<td>89672</td>
<td>3631</td>
</tr>
<tr>
<td>55-60</td>
<td>86041</td>
<td>5341</td>
</tr>
<tr>
<td>60-65</td>
<td>80700</td>
<td>7171</td>
</tr>
<tr>
<td>65-70</td>
<td>73529</td>
<td>9480</td>
</tr>
<tr>
<td>70-75</td>
<td>66049</td>
<td>11562</td>
</tr>
<tr>
<td>75-80</td>
<td>52487</td>
<td>14192</td>
</tr>
<tr>
<td>80-85</td>
<td>38295</td>
<td>14752</td>
</tr>
<tr>
<td>85+</td>
<td>23543</td>
<td>23543</td>
</tr>
</tbody>
</table>
Example 6. Table 1 is a part of a life table for the 1970 California, USA, population. Column (1) shows the age intervals in years. Column (2) is the number of (life table) people living at the beginning of each age interval. Thus, the column shows that there are 100,000 (life table) people alive at the exact age 0 (that is, the population size at birth); of these 98,199 survive to the exact age of 1 year (the first birthday), 97,883 survived to the exact age of 5 years, etc., and finally 23,543 survived to the exact age of 85 years. Each figure in column (3) is the number of people dying within the corresponding age interval. Among the 100,000 living at age 0, 1801 died during the age interval (0,1), 316 died between ages 1 and 5, etc., and 23543 died beyond age 85 years.

For the purpose of illustration, we consider 100,000 newborns who are subject to the mortality experience of the 1970 California population. What is the probability that a newborn will survive to his first birthday? In this example, the "random experiment" is the baby's first year of life; possible outcomes are survival or death of the 100,000 infants; the event A of interest is a newborn's survival to his first birthday. Since 98,199 of the 100,000 newborns (the possible number of survivors) actually survived (event A occurred), the probability that a newborn will survive to his first birthday is

$$\frac{n(A)}{n} = \frac{98,199}{100,000} = .98199 \text{ or 981.99 per 1,000}$$

Similarly, the probability that a newborn will survive to the fifth birthday is 97883/100,000 = .97883, to the 10th birthday is 97699/100,000 = .97699.

For the probability of death, we use the corresponding number of deaths in the numerator of the formula. Thus we have
Pr{a newborn will die in the first year of life} = \frac{1801}{100,000} = .01801 or 18.01 per 1,000

and

Pr{a newborn will die in interval (1,5)} = \frac{316}{100,000} = .00316 or 3.16 per 1,000

2.19.1. Conditional Probability. The probabilities computed above are absolute probabilities based on the 100,000 live births. When the base population is changed, we have conditional probabilities:

Pr{a child alive at age 1 will die in interval (1,5)}

= Pr{a child will die in interval (1,5) | he is alive at age 1}

= \frac{\text{number dying in (1,5)}}{\text{number living at age 1}} = \frac{316}{98199} = .00322 or 3.22 per 1,000

and

Pr{a child alive at age 5 will die in interval (5,10)}

= \frac{\text{number dying in (5,10)}}{\text{number living at age 5}} = \frac{184}{97883} = .00188 or 18.8 per 1,000

These conditional probabilities, which are based on the number of individuals living at the beginning of the corresponding age interval, are known as the age-specific probabilities of dying. Other conditional probabilities are possible, depending upon the given condition and the event of interest. The following are a few examples:

Pr{an individual of age 25 will survive to age 50}

= \frac{\text{number living at age 50}}{\text{number living at age 25}} = \frac{89672}{96216} = .93199

and
Pr\{an individual of age 25 will die before age 50\}

\[
\frac{\text{number dying between ages 25 and 50}}{\text{number living at age 25}} = \frac{96216 - 89672}{96216}
\]

\[
= \frac{6544}{96216} = .06801
\]

where the number 6544 can be determined also from the number of deaths in all the intervals from 25 to 50:

\[
6544 = 681 + 766 + 1060 + 1583 + 2454.
\]

Since an individual alive at age 25 will either survive to age 50 or die before age 50, the corresponding probabilities must add to unity:

\[
.93199 + .06801 = 1.00000
\]

For an individual alive at age 20, the corresponding probabilities are:

Pr\{an individual of age 20 will survive to age 45}\]

\[
= \frac{92126}{96966} = .95009
\]

and

Pr\{an individual of age 20 will die before age 45\} = 1 - .95009 = .04991.

2.19.2. Probabilities of Composite Events. Let A be an event that a male of age 25 survives to age 50 and \(\bar{A}\) he dies before age 50; let B be an event that a female of age 20 survives to age 45 and \(\bar{B}\) she dies before age 45. If they are subject to the probability of dying shown in the above table and if their survival is independent of one another, then we can use the multiplication theorem to compute the following probabilities:
Pr{both male and female live for 25 years}

= Pr{A and B} = Pr{A} x Pr{B} = .93199 x .95008 = .88547,

Pr{both die within 25 years}

= Pr{\overline{A} and \overline{B}} = Pr{\overline{A}} x Pr{\overline{B}} = .06801 x .04991 = .00339

Pr{male lives and female dies in 25 years}

= Pr{A and \overline{B}} = Pr{A} x Pr{\overline{B}} = .93199 x .04991 = .04652

and

Pr{male dies and female lives for 25 years}

= Pr{\overline{A} and B} = Pr{\overline{A}} x Pr{B} = .06801 x .95009 = .06462.

Since either both male and female will survive a period of 25 years, or one of them dies, or both die, the sum of the above probabilities is equal to one:

.88547 + .00339 + .04652 + .06462 = 1.

The reader may wish to compute similar probabilities for other ages or for a period different from 25 years.

2.19.3. Probability of Dissolution of Marriage. The above probabilities can be used to compute joint life insurance premiums or dissolution of marriages. For example, if a husband is of age 25 and his wife of age 20, the probability that their marriage will be dissolved in 25 years due to death may be computed as follows:

Pr{dissolution of marriage in 25 years due to death}

= Pr{one or both of them die in 25 years}

= Pr{(A and \overline{B}) or (\overline{A} and B) or (\overline{A} and \overline{B})}
Here the three events \((A \text{ and } B)\), \((A \text{ and } \overline{B})\), and \((A \text{ and } \overline{B})\) are mutually exclusive; we use the addition theorem and the above numerical values to obtain the probability

\[
\Pr((A \text{ and } B)) + \Pr((A \text{ and } \overline{B})) + \Pr((A \text{ and } \overline{B}))
\]

\[
= .04652 + .06462 + .00339 = .11453 .
\]

Thus the probability of dissolution of their marriage is better than 10 percent. On the other hand,

\[
\Pr(\text{their marriage will not be dissolved in 25 years})
\]

\[
= \Pr(\text{both live for 25 years}) = \Pr(A \text{ and } B) = .88547
\]

Obviously, the two probabilities are complementary to each other, and

\[
.11453 + .88547 = 1.00000 .
\]
CHAPTER 2

DEATH RATES AND ADJUSTMENT OF RATES

1. Age Specific Death Rates

For a specific age interval \((x_i, x_{i+1})\), the death rate, \(M_i\), is defined as follows:

\[
M_i = \frac{\text{Number dying in } (x_i, x_{i+1})}{\text{Number of years lived in } (x_i, x_{i+1}) \text{ by those alive at } x_i}
\] (1.1)

Suppose that of \(\ell_i\) people living at exact age \(x_i\), \(d_i\) die between age \(x_i\) and \(x_{i+1}\) and each of \(d_i\) people lives on the average a fraction, \(a_i\), of the interval \((x_i, x_{i+1})\). Then the death rate \(M_i\) defined in (1.1) may be expressed in the formula

\[
M_i = \frac{d_i}{n_i(\ell_i - d_i) + a_i \cdot n_i \cdot d_i}
\] (1.2)

where \(n_i = x_{i+1} - x_i\) is the length of the interval \((x_i, x_{i+1})\), \(n_i(\ell_i - d_i)\) is the number of years lived in \((x_i, x_{i+1})\) by the \((\ell_i - d_i)\) survivors, and \(a_i \cdot n_i \cdot d_i\) is the number of years lived by the \(d_i\) people who die in the interval. The unit of a death rate is the number of deaths per person-years. The corresponding estimate of probability of dying, given by

\[
\hat{q}_i = \frac{d_i}{\ell_i}
\] (1.3)

is a pure number. From (1.2) and (1.3), we find a relationship between \(q_i\) and \(M_i\)

\[
\hat{q}_i = \frac{n_iM_i}{1 + (1 - a_i)n_iM_i}
\] (1.4)

We see then that the age-specific death rate and the probability of dying are two different concepts and they are related by formula (1.4). Consider as an example the age interval \((1, 5)\) in the 1970 California life table population. Here \(x_1=1, x_5=5,\) and \(n_1=5-1=4\). From Section 2, Table 1, we find \(\hat{q}_1 = 98199\),
$d_1 = 316$, and from Appendix [V], $a_1 = .41$. The death rate is

$$M_1 = \frac{316}{4(98199-316) + .41 \times 4 \times 316}$$

$$= .000806$$

and the estimate of the probability is

$$\hat{q}_1 = \frac{316}{98199} = .00322$$

Formula (1.2) of the age-specific death rate is expressed in terms of a life table framework where $l_i$ people are followed for $n_i$ years to determine the number of deaths ($d_i$) and the number of survivors ($l_i - d_i$) at the end of $n_i$ years. In a current population, such as the 1970 California population, an age specific death rate is computed from the mortality and population data during a calendar year (1970). Instead of $d_i$ defined in a life table, we have $D_i$, the observed number of deaths occurring to people in the age group $(x_i, x_{i+1})$ during a calendar year. To derive a formula for the death rate as in (1.2), we let $N_i$ be the (hypothetical) number of people alive at exact age $x_i$; among them $D_i$ deaths occur. Then we have the death rate

$$M_i = \frac{D_i}{n_i(N_i - D_i) + a_i n_i D_i}$$  \hspace{1cm} (1.2a)$$

and an estimate of the probability $q_i$,

$$\hat{q}_i = \frac{D_i}{N_i}$$  \hspace{1cm} (1.3a)$$

They also have the relationship in (1.4).

Since $N_i$ is a hypothetical number, the denominator of (1.2a) and the death rate for a current population cannot be computed from (1.2a). Customarily,
the denominator of (1.2a) is estimated by the midyear (calendar year) population $P_i$ for age group $(x_i, x_{i+1})$, and hence the age-specific death rate is given by

$$M_i = \frac{D_i}{P_i}.$$ (1.5)

Although it is a well known and accepted definition of age-specific death rates, formula (1.5) is much more meaningful when $P_i$ is interpreted as an estimate of the denominator in (1.1).

In California 1970, there were $D_1 = 1049$ deaths occurring in age interval $(1,5)$, and $P_1 = 1,302,198$ people of ages 1 to 5 at midyear. Therefore, the corresponding death rate is

$$M_1 = \frac{D_1}{P_1} = \frac{1049}{1,302,198} = .000806.$$ (1.5)

A death rate usually is a small number; its significance is not easily appreciated. To remedy this, the numerical value of a death rate is multiplied by a number, such as 1000, which is called the base. The formula of a death rate often appears as

$$M_1 = \frac{D_1}{P_1} \times 1000.$$ (1.5a)

Thus, instead of $M_1 = .000806$, we have $M_1 = .806$ per 1000 person-years.\(^1\)

It should be clear that in formula (1.5) and (1.5a) the number of deaths $D_i$ in the numerator and the midyear population $P_i$ in the denominator refer to the same population, such as the 1970 California population between ages 1 and 5. The population and the base must be clearly stated in a death rate. For example, the death rate for the age group 1 to 5 years in the 1970 California population is .806 per 1000.

\(^1\)The words "person-years" are often deleted.
When a death rate is for an entire life, it is called the crude death rate. In formula:

\[ M = \frac{D}{P} \times 1000 \]  

where

\[ D = \sum D_i \]

is the total number of deaths occurring during a calendar year, and

\[ P = \sum P_i \]

is the total midyear population of a community, or a country, in question.

Death rates may be computed for any specific category of people in a population. Sex-specific death rates, occupation-specific death rates, age-sex-specific death rates, are examples. In each case, the specific rate is defined as the number of deaths occurring to people in the stated category during a calendar year divided by the midyear population of the same category.

Death rates may also be computed for specific causes such as death rates from cancer, tuberculosis, or heart diseases. These are known as cause-specific death rates. Here it is deaths, rather than population, that is divided into categories. A cause-specific death rate is defined as the number of deaths from the specific cause divided by the midyear population. In formula, the death rate from cause \( R_\delta \) is given by:

\[ M_\delta = \frac{D_\delta}{P} \times 100,000 \]  

Here \( D_\delta \) is the number of deaths from cause \( R_\delta \) during a calendar year in question, the base is 100,000 because of the small magnitude of the rate.
Prevalence of diseases varies with age. Cardiovascular disease, for example, is more prevalent among the aged than among young people; the converse is true for infectious diseases. Therefore, age-cause-specific death rates are in common use. For the age interval \((x_i, x_{i+1})\) and cause \(R_0\), the specific death rate, \(M_{i0}\), is computed from

\[
M_{i0} = \frac{D_{i0}}{P_i} \times 100,000,
\]

where \(D_{i0}\) is the number of deaths from cause \(R_0\) occurring to people in age group \((x_i, x_{i+1})\) during a calendar year, and \(P_i\) is the midyear population of the same age group. Here a base, 100,000, is used.

2. **Infant Mortality**

In the human population, mortality is the highest among newborns and among the elderly. Infant mortality also has a great impact on the population distribution in later years of life. Various efforts have been made in different countries to reduce infant deaths, and many of these efforts have resulted in a considerable amount of success. Mortality in the first year of life has been decreasing, especially in the developed countries prior to 1950. Since many different causes affect mortality from conception to the end of the first year of life, this period of human life has been divided into subintervals and designated by special names, as shown in the following table.
Table 1. Fetal death and infant mortality

<table>
<thead>
<tr>
<th>Designation</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early fetal death</td>
<td>Under 20 weeks of gestation</td>
</tr>
<tr>
<td>Intermediate fetal death</td>
<td>20-27 weeks of gestation</td>
</tr>
<tr>
<td>Late fetal death</td>
<td>28 or more weeks of gestation</td>
</tr>
<tr>
<td>Neonatal death</td>
<td>Under 28 days of age</td>
</tr>
<tr>
<td>Post neonatal death</td>
<td>28 days to end of first year of life</td>
</tr>
<tr>
<td>Infant death</td>
<td>Under one year of age</td>
</tr>
</tbody>
</table>

The corresponding definitions of death rates differ somewhat from the definition of the age-specific death rate discussed in the preceding section. The following rates are measures of mortality for a defined population during a given calendar year:

2.1. Fetal death rate (alias "stillbirth rate"). Two definitions are available:

\[
\text{Number of fetal deaths of 28 or more weeks of gestation} \times 1000
\]

\[
\text{Number of live births + fetal deaths of 28 or more weeks of gestation}
\]

(2.1)

\[
\text{Number of fetal deaths of 20 or more weeks of gestation} \times 1000
\]

\[
\text{Number of live births + fetal deaths of 20 or more weeks of gestation}
\]

(2.2)

2.2. Neonatal mortality rate.

\[
\text{Number of deaths under 28 days of age} \times 1000
\]

\[
\text{Number of live births}
\]

(2.3)

2.3. Perinatal mortality rate. There are two definitions in common use:

\[
\text{Number of deaths under 7 days + fetal deaths of 28 or more weeks of gestation}
\]

\[
\text{Number of live births + fetal deaths of 28 or more weeks of gestation}
\]

(2.4)
Number of deaths under 28 days of life + fetal deaths of 20 or more weeks of gestation \(\times 1000\)

\[
\frac{\text{Number of live births} + \text{fetal deaths of 20 or more weeks of gestation}}{\text{Number of live births}} \times 1000
\]

(2.5)

The second definition covers a longer period both in gestation and after birth.

2.4. Post neonatal mortality rate.

\[
\frac{\text{Number of deaths at age 28 days through one year}}{\text{Number of live births} - \text{neonatal deaths}} \times 1000
\]

(2.6)

It is incorrect not to subtract neonatal deaths from live births in the denominator.

Difference in numerical value due to this error depends on neonatal mortality; the difference may be considerable when neonatal mortality is high.

2.5. Infant mortality rate.

\[
\frac{\text{Number of deaths under one year of age}}{\text{Number of live births}} \times 1000
\]

(2.7)

Mortality rates defined above are closer to probability than to age-specific death rates, since in each instance the numerator is a part of the denominator.

There are measures of mortality which resemble neither probability nor age-specific death rates. Nevertheless, they are quite useful in mortality analysis. Some examples follow.

2.6. Fetal death ratio.

\[
\frac{\text{Number of fetal deaths of 20 or more weeks of gestation}}{\text{Number of live births}} \times 1000
\]

(2.8)

2.7. Maternal mortality rate.

\[
\frac{\text{Number of maternal deaths}}{\text{Number of live births}} \times 1000
\]

(2.9)

A maternal death is a death occurring to women due to complications of pregnancy, childbirth and the puerperium (period after delivery). While
not strictly a measure of risk, the maternal mortality rate indicates a "price" in terms of mother's life that a human population pays for every infant brought into the world.

It was indicated at the beginning of this section that fetal death and infant mortality have experienced a constant decline. We shall now substantiate this statement by citing a report prepared by Helen C. Chase in 1967. She states:

"One of the notable health accomplishments in the 20th century has been the decline in infant mortality. Over the first half of the century the rapid decline in mortality among infants became an accepted component of the nation's health. In the past decade, it has become difficult to adjust to the idea that infant mortality in the United States is no longer declining at its former rate."

The deceleration of the rate of decline in infant mortality, however, was not peculiar to the United States. Similar changes in trend have appeared in several European countries. Tables 2 and 3 summarize these findings. It may be noted that even during the period from 1950 to 1962, the reduction in fetal death and infant mortality was still substantial. Table 4 shows the fetal and infant mortality in the United States from 1960 to 1970. The reductions in all categories are still quite considerable.

Table 2. Infant mortality rates and percent reduction: Selected countries, 1935, 1950, and 1962

<table>
<thead>
<tr>
<th>Country</th>
<th>Infant Mortality Rate</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denmark</td>
<td>71.0</td>
<td>30.7</td>
</tr>
<tr>
<td>England &amp; Wales</td>
<td>56.9</td>
<td>29.9</td>
</tr>
<tr>
<td>Netherlands</td>
<td>40.0</td>
<td>26.7</td>
</tr>
<tr>
<td>Norway</td>
<td>44.4</td>
<td>28.8</td>
</tr>
<tr>
<td>Scotland</td>
<td>76.8</td>
<td>37.6</td>
</tr>
<tr>
<td>Sweden</td>
<td>45.9</td>
<td>21.0</td>
</tr>
<tr>
<td>United States</td>
<td>55.7</td>
<td>29.2</td>
</tr>
</tbody>
</table>

Rates per 1,000 live births.

Table 3. Fetal mortality rates* and percent reduction, selected countries, 1955 and 1963

<table>
<thead>
<tr>
<th>Country</th>
<th>Fetal Mortality Rates</th>
<th>Difference</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1955</td>
<td>1963</td>
<td></td>
</tr>
<tr>
<td>Denmark</td>
<td>17.9</td>
<td>11.4</td>
<td>6.5</td>
</tr>
<tr>
<td>England &amp; Wales</td>
<td>23.2</td>
<td>17.2</td>
<td>6.0</td>
</tr>
<tr>
<td>Netherlands</td>
<td>17.0</td>
<td>14.3</td>
<td>2.7</td>
</tr>
<tr>
<td>Norway</td>
<td>14.9</td>
<td>12.6</td>
<td>2.3</td>
</tr>
<tr>
<td>Scotland</td>
<td>24.6</td>
<td>19.1</td>
<td>5.5</td>
</tr>
<tr>
<td>Sweden</td>
<td>16.7</td>
<td>12.0</td>
<td>4.7</td>
</tr>
<tr>
<td>United States</td>
<td>12.6</td>
<td>11.3</td>
<td>1.3</td>
</tr>
</tbody>
</table>

*Fetal deaths of 28 or more weeks of gestation. Rates per 1,000

SOURCE: Helen C. Chase, ibid

Table 4. Fetal and infant mortality and percent reduction, United States, 1960 and 1970

<table>
<thead>
<tr>
<th></th>
<th>1960</th>
<th>1970</th>
<th>Difference</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fetal death rate (20 weeks + gestation)</td>
<td>15.8</td>
<td>14.0</td>
<td>1.8</td>
<td>11.4</td>
</tr>
<tr>
<td>Neonatal mortality rate</td>
<td>18.7</td>
<td>15.1</td>
<td>3.6</td>
<td>19.3</td>
</tr>
<tr>
<td>Postneonatal mortality rate</td>
<td>7.5</td>
<td>4.9</td>
<td>2.6</td>
<td>34.7</td>
</tr>
<tr>
<td>Infant mortality rate</td>
<td>26.0</td>
<td>20.0</td>
<td>6.0</td>
<td>23.1</td>
</tr>
<tr>
<td>Fetal death ratio</td>
<td>16.1</td>
<td>14.2</td>
<td>1.9</td>
<td>11.8</td>
</tr>
<tr>
<td>Maternal mortality rate (per 100,000)</td>
<td>37.1</td>
<td>21.5</td>
<td>15.6</td>
<td>42.0</td>
</tr>
</tbody>
</table>

Rates per 1,000
3. Adjustment of Rates

Specific death rates presented in Section 3 are essential in mortality analysis. Individually, these rates describe mortality experience within respective categories of people. Collectively, they represent a mortality pattern of the population in question. When a collective measure of mortality of an entire population is required, specific rates provide the fundamental components. One of the central tasks in statistical analysis of mortality data is making comparisons of experiences of various communities or countries; summarization of specific rates in a single number is extremely important. Since age-sex distribution varies from one community to another, and from one country to another, adjustment for such variation will have to be made in summarizing specific rates. The resulting single figure is called the adjusted rate. Adjustment can be made with respect to age, sex, occupation and possibly others. For simplicity, we shall consider only age-adjusted rates. Adjusted rates for other variables, such as sex-adjusted rates, age-sex-adjusted rates, etc., can be computed similarly. Various methods of adjustment have been proposed; some of these are listed in Table 5. It is the purpose of this section to review them. But first, let us introduce some notations.

In the adjustment of rates, two populations are usually involved:
A community, \( u \), during a calendar year (the population of interest) and a standard population, \( s \). For each age interval \((x_i, x_{i+1})\) in the community, \( u \), let \( D_{ui} \) be the number of deaths; \( P_{ui} \) the midyear population; \( M_{ui} \), its specific death rate; and let \( n_i = x_{i+1} - x_i \) be the length of the interval.

The sum

\[
\sum_{i} D_{ui} = D_u
\]

(3.1)

is the total number of deaths occurring in the community during the
is the total midyear population. For the standard population, the symbols
\( D_s, P_s, M_s, D_s, P_s \) are defined similarly. These symbols are similar
to those used in Section 1 except for the addition of the subscripts \( u \) and \( s \).

Table 5. Age-adjusted death rates and mortality indices

<table>
<thead>
<tr>
<th>Title</th>
<th>Formula</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude death rate (C.D.R.)</td>
<td>( \frac{\Sigma P_{ui} M_{ui}}{P_u} )</td>
<td>Linder, F. E. and Grove, R. D. (1943)</td>
</tr>
<tr>
<td>Direct method of adjustment (D.M.D.R.)</td>
<td>( \frac{P_s M_{ui}}{P_s} )</td>
<td>&quot;The Registrar General's Statistical Reviews of England &amp; Wales for the Year 1934&quot;</td>
</tr>
<tr>
<td>Comparative mortality rate (C.M.R.)</td>
<td>( \frac{1}{2} \Sigma \left( \frac{P_{ui}}{P_u} + \frac{P_{si}}{P_s} \right) M_{ui} )</td>
<td>Ibid</td>
</tr>
<tr>
<td>Indirect method of adjustment (I.M.D.R.)</td>
<td>( \frac{D_s / P_s (D_u / P_u)}{P_{ui} M_{ui} / P_u} )</td>
<td>&quot;The Registrar General's Decennial Supplement, England and Wales, 1921, Part III.&quot;</td>
</tr>
<tr>
<td>Life table death rate (L.T.D.R.)</td>
<td>( \frac{I_{L_{ui}}}{I_{L_i}} )</td>
<td>Brownlee, J. (1913) (1922)</td>
</tr>
<tr>
<td>Equivalent average death rate (E.A.D.R.)</td>
<td>( \frac{\ln_{M_{ui}}}{\ln_{i}} )</td>
<td>Yule, G. U. (1934)</td>
</tr>
<tr>
<td>Relative mortality index (R.M.I.)</td>
<td>( \frac{P_{ui} M_{ui}}{P_u M_{si}} )</td>
<td>Linder, F. E. and Grove, R. D. (1943)</td>
</tr>
<tr>
<td>Mortality index (M.I.)</td>
<td>( \frac{\ln_{i} M_{ui}}{\ln_{i} M_{si}} )</td>
<td>Yerushalmy, J. (1951)</td>
</tr>
<tr>
<td>Standardized mortality ratio (S.M.R.)</td>
<td>( \frac{P_{ui} M_{ui}}{P_{ui} M_{si}} )</td>
<td>&quot;The Registrar General's Statistical Review of England and Wales, 1958&quot;</td>
</tr>
</tbody>
</table>
3.1. Crude death rate. As was mentioned in Section 1, the crude death rate is the ratio of the total number of deaths occurring in a community during a calendar year to the community's total midyear population:

$$\text{C.D.R.} = \frac{D}{P}$$  \hspace{1cm} (3.3)

The crude death rate, which is the most commonly used and conveniently computed single value, bears a close relationship to age-specific death rates. The numerator in (3.3) is the sum of the number of deaths occurring in all age categories:

$$D = \sum_{i} D_{ui}$$  \hspace{1cm} (3.4)

By definition, the age-specific death rate for age interval \((x_i, x_{i+1})\) is given by

$$M_{ui} = \frac{D_{ui}}{P_{ui}}$$  \hspace{1cm} (3.5)

so that the number of deaths \((D_{ui})\) is the product of the age-specific death rate \((M_{ui})\) and the corresponding midyear population \((P_{ui})\):

$$D_{ui} = P_{ui} M_{ui}$$  \hspace{1cm} (3.6)

Therefore, the total number of deaths in (3.4) may be rewritten as

$$D = \sum_{i} P_{ui} M_{ui}$$  \hspace{1cm} (3.7)

Substituting (3.7) in (3.3) yields

$$\text{C.D.R.} = \sum_{i} \frac{P_{ui}}{P} M_{ui}$$  \hspace{1cm} (3.8)

where the summation is taken over the entire life span. Thus the C.D.R. is a weighted mean of age-specific death rates with the actual population
proportions $P_{ui}/P_u$ experiencing the mortality used as weights. From this viewpoint, the C.D.R. is the most meaningful single figure summarizing the mortality experience of a given population.

The C.D.R., however, is not without deficiencies. The quantity on the right-hand side of (3.8) is a function of both the age-specific death rates and the age-specific population proportions. As a weighted mean of age-specific death rates, the C.D.R. is affected by the population composition of the community in question. This disadvantage becomes apparent when the C.D.R. is used as a common measure to compare the mortality experience of several communities. The example in Table 6 illustrates this point.

Table 6. Age-specific death rates and crude death rates for communities A and B.

<table>
<thead>
<tr>
<th></th>
<th>Community A</th>
<th></th>
<th>Community B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Population</td>
<td>Deaths per 1000</td>
<td>Population</td>
<td>Deaths per 1000</td>
</tr>
<tr>
<td>Children</td>
<td>10,000</td>
<td>80</td>
<td>25,000</td>
<td>250</td>
</tr>
<tr>
<td>Adults</td>
<td>15,000</td>
<td>165</td>
<td>15,000</td>
<td>180</td>
</tr>
<tr>
<td>Senior citizens</td>
<td>25,000</td>
<td>375</td>
<td>10,000</td>
<td>160</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>50,000</strong></td>
<td><strong>620</strong></td>
<td><strong>50,000</strong></td>
<td><strong>590</strong></td>
</tr>
</tbody>
</table>

Although the age-specific death rate for each age group in Community A is lower than that for the corresponding age group in Community B, the crude death rate for Community A, (12.4), is higher than that for Community B.
This inconsistency is explained by differences in the population composition of the two communities. Community A consists of a larger percentage of older people, who are subject to a high mortality and contribute more deaths. As a result, Community A's overall crude death rate is higher than that of the more youthful Community B.

3.2. Direct Method Death Rate (D.M.D.R.). One way of adjusting for peculiarities of population composition is to introduce a standard population common to all the communities. When the age-specific death rates of a community are applied to such a standard population, we obtain a death rate adjusted by the direct method:

$$D.M.D.R. = \frac{\sum_{i} p_{si} M_{ui}}{P_{s}}$$

(3.9)

The D.M.D.R. is thus a weighted mean of the age-specific death rates $M_{ui}$ of a community with standard population proportions, $P_{si}/P_{s}$, applied as weights. If formula (3.9) is rewritten as

$$D.M.D.R. = \frac{\sum_{i} p_{si} M_{ui}}{P_{s}}$$

(3.10)

the numerator becomes the number of deaths that would occur in the standard population if it were subject to the age-specific rates of the community. The ratio of the total "expected deaths" to the entire standard population yields the D.M.D.R. However, the D.M.D.R., as well as other age adjusted rates which follow, is not designed to measure the mortality experience of a community. It is simply a means for evaluating mortality experience of one community relative to another. An age-adjusted rate should be considered with this understanding.

Computation of the D.M.D.R. based on the example in Table 6 is given in Table 7. In this illustration, the combined population of the two
communities is used as the standard population shown in column (1) in Table 7. The age-specific rates in the two communities are recorded in columns (2) and (3), respectively. Each of the specific rates is then applied to the standard population in the same age group to obtain the number of deaths expected in the standard population shown in columns (4) and (5). Summing these expected numbers of deaths over all age groups yields the total number of deaths, 1,135 and 1,270, respectively. When the total number of deaths is divided by the total standard population, we obtain the D.M.D.R.

Table 7. Direct method age-adjusted rates for Communities A and B

<table>
<thead>
<tr>
<th>Standard Population</th>
<th>Age Specific Rates</th>
<th>Expected No. of Deaths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Community A</td>
<td>Community B</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>35,000</td>
<td>8.0</td>
<td>10.0</td>
</tr>
<tr>
<td>30,000</td>
<td>11.0</td>
<td>12.0</td>
</tr>
<tr>
<td>35,000</td>
<td>15.0</td>
<td>16.0</td>
</tr>
<tr>
<td>100,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Adjusted Rate: Community A = 11.35/1,000
                Community B = 12.70/1,000

Using a single standard population, the direct method of adjustment eliminates the effect of differences in age-composition of the communities under study; the result nevertheless depends upon the composition of the population selected as a standard. When communities with very different mortality patterns are compared, different standard populations may even produce contradictory results. In computing the age-adjusted rate for the 1940 white male population of Louisiana and New Mexico, Yerushalmy (1951)
found that the age-adjusted rate for Louisiana (13.06 per 1,000) was slightly higher than the rate for New Mexico (13.05 per 1,000) when the 1940 U.S. population was used as the standard; but the rate for Louisiana (10.14 per 1,000) was lower than the rate for New Mexico (11.68 per 1,000) when the 1901 population of England and Wales was used as the standard. This kind of dilemma has led to the development of other methods of adjustment.

3.3. Comparative Mortality Rate (C.M.R.). In this method of adjustment, both the age composition of the community and that of the standard population are taken into account. The formula is

\[ C.M.R. = \frac{1}{2} \sum_i \left( \frac{P_{ui}}{P_i} + \frac{P_{si}}{P_s} \right) M_{ui} \]  

(3.11)

Easy computations show that the first sum is the crude death rate of the community,

\[ \sum_i \frac{P_{ui}}{P_i} M_{ui} = \sum_i \frac{D_{ui}}{P_i} = \frac{D_u}{P_u} , \]

while the second sum is the direct method death rate. Thus the C.M.R. is simply the mean of the C.D.R. and D.M.D.R. Using the previous example once again, we find

C.M.R. (community A) = \frac{1}{2}(12.4 + 11.35) = 11.87

C.M.R. (community B) = \frac{1}{2}(11.8 + 12.70) = 12.25
3.4. **Indirect Method Death Rate (I.M.D.R.).** In the age-adjusted rate by the indirect method, the crude death rate of the community is multiplied by the ratio of the crude death rate of a standard population to the death rate that would be expected in the standard population if it had the same composition as the given community.\(^1\) The formula for the I.M.D.R. is

\[
\text{I.M.D.R.} = \frac{\frac{D_s}{P_s} \frac{D_u}{P_u}}{\sum_{i} \frac{P_u M_s}{P_i} / \frac{P_u}{P_i}}
\]

The denominator of the first factor in (3.12)

\[
\frac{\sum_{i} P_u M_s}{\sum_{i} P_u / P_i}
\]

is in effect a D.M.D.R. when the position of a community and a standard population is interchanged: the age-specific death rates of a standard population \(M_s\) are applied to a community population \(P_u\).

When the population composition of a community and a standard population are the same, so that

\[
\frac{P_u M_s}{P_u / P_s} = \frac{P_u M_s}{P_u / P_i}
\]

for every interval \((x_i, x_{i+1})\), then the first factor in (3.12) becomes unity,

\[
\frac{\sum_{i} P_u M_s}{P_u / P_i} = \frac{\sum_{i} P_u M_s}{P_u / P_i} = 1
\]

and the I.M.D.R. is equal to the C.D.R. of the community. If a community should have a higher proportion of old people than the standard population, then for the old age group

\(^1\)A method suggested by Herald Westergaard is also used in the study of death rates. Westergaard's formula, however, can be derived from the indirect method and vice versa.
and the crude death rate of the community will be greater than the I.M.D.R.

Formula (3.12) can be written also as

$$I.M.D.R. = \frac{D_s}{\sum_i w_i s} \frac{D_u}{P}$$

$$= \frac{D_s}{\sum_i w_i s} \frac{\sum_i P M_i}{P}$$

$$= \sum_i w_i M_i$$

(3.13)

where

$$w_i = \frac{D_s}{\sum_i w_i s} \frac{P}{P}$$

(3.14)

Here the weights $w_i$ do not add to unity unless the community and the standard population have the same composition. Therefore, generally the I.M.D.R. is not an average of the specific death rates, and is not directly comparable with the C.D.R. or the D.M.D.R.

One advantage of the indirect method of adjustment may be noted. Since only the total number of deaths in a community ($D_u$) is in the formula, this method of adjustment requires less information from a community than the direct method.

3.5. **Life Table Death Rate (L.T.D.R.)**. Most of the methods of adjustment rely on a standard population or its rates. One exception is the L.T.D.R. which is defined as
where $L_i$ is the number of years spent in $(x_i, x_{i+1})$ by a life table population and $T_0 = L_0 + L_1 + \ldots$ (3.16)

A full appreciation of this method of adjustment requires the knowledge of the life table discussed in Chapter 5; a brief discussion of formula (3.15) follows. Given $l_0$ people alive at age 0 who are subject to the age-specific death rates of the community, $L_i/T_0$ is the proportion of their life time spent in the age interval $(x_i, x_{i+1})$. In other words, the L.T.D.R. shown in formula (3.15) is a weighted mean of the age specific death rates ($M_{ui}$) with the proportion of life time spent in $(x_i, x_{i+1})$ being used as weights. Since the weights $L_i/T_0$ depend solely on the age-specific death rates, the L.T.D.R. is independent of the population composition either of a community or a standard population.

As we will see in Chapter 5, the age specific death rate $M_{ui}$ is equal to the ratio $d_i/L_i$,

$$M_{ui} = \frac{d_i}{L_i}$$

hence

$$L_i M_{ui} = d_i$$ (3.17)

where $d_i$ is the life table deaths in age interval $(x_i, x_{i+1})$. The sum,

$$d_0 + d_1 + \ldots = l_0$$ (3.18)

is equal to the total number of individuals $l_0$ at age 0. Substituting (3.17) in (3.15) and recognizing (3.18), we have
The inverse

\[
T_0 / \ell_0 = e_0
\]

is known as the (observed) expectation of life at age 0; therefore

\[
L.T.D.R. = \frac{1}{e_0}
\]

3.6. Equivalent Average Death Rate (E.A.D.R.). In this method of adjustment each age-specific rate is weighted with the corresponding interval length rather than the number of people for which the rate is computed. In formula, it is:

\[
E.A.D.R. = \sum \frac{n_i}{\hat{M}_{ui}}
\]

where \( n_i = x_{i+1} - x_i \). The last age interval is an open interval, such as 60 and over, and the corresponding death rate is usually high. An upper limit must be set for the last interval in order to prevent the high death rate of the elderly from asserting an undue effect on the resulting adjusted rate. G. U. Yule, the original author of the index, suggested that the limit of the last age interval be set at 65 years. It may be observed that since there are fewer people in the old age group, the E.A.D.R. places more emphasis on old ages than the C.R.D. or the D.M.D.R.
3.7. Relative Mortality Index (R.M.I.). The basic quantities used in the relative mortality index are the ratios of specific rates of a community to the corresponding rates of a standard population. The index is a weighted mean of these ratios, obtained by using the community age-specific population proportions as weights. The formula for the R.M.I. is

$$R.M.I. = \sum_{i} \frac{P_{ui}}{P_{u}} \frac{M_{ui}}{M_{si}} \quad (3.23)$$

The R.M.I. strongly reflects the mortality pattern of young age groups where small changes in the specific rates may produce large differences in the value of the index.

When (3.23) is rewritten as

$$R.M.I. = \frac{1}{P_{u}} \sum_{i} \frac{D_{ui}}{M_{si}}$$

we see that the R.M.I. may be computed without knowledge of the community's population by age.

3.8. Mortality Index (M.I.). This index is also a weighted average of the ratios of community age-specific death rates to the corresponding rates of a standard population. It differs from the relative mortality index in that the weights used here are the lengths of age intervals. The formula for the index is

$$M.I. = \frac{\sum_{i} M_{ui}}{\sum_{i} n_{i} M_{si}} \quad (3.24)$$

Generally, the M.I. is affected more by the death rates in old age groups than is the R.M.I. A main feature of this method is that, for intervals of the same length, a constant change of the ratio $M_{ui}/M_{si}$ has an equal effect on the value of the index.
3.9. **Standardized Mortality Ratio (S.M.R.).** The General Register Office of Great Britain has used the S.M.R. in the *Statistical Review of England and Wales* since 1958. It is a ratio of the number of deaths occurring in a community to the expected number of deaths in the community if it were subject to the age-specific rates of a standard population. In formula,

\[
S.M.R. = \frac{\sum D_{ui}}{\sum P_{ui} M_{si}} = \frac{\sum P_{ui} M_{ui}}{\sum P_{ui} M_{si}}
\]

(3.25)

Since the numerator is the total deaths in the community, (3.25) can be rewritten as

\[
S.M.R. = \frac{D_u}{\sum P_{ui} M_{si}}
\]

(3.26)

or

\[
S.M.R. = \frac{D_u / P_u}{\sum M_{si} / P_u}
\]

(3.27)

Thus, the S.M.R. is the crude death rate of a community divided by the direct method death rate, when standard population age-specific death rates are applied to a community population.
CHAPTER 3

STANDARD ERROR OF MORTALITY RATES

1. Introduction

An age-specific death rate is a measure of the mortality experience of a defined population group over a given period of time. An age-adjusted death rate, as a function of age-specific rates, is designed to summarize the mortality experience of an entire population for the purpose of comparing it with that of other populations. As with any observable statistical quantity, both the specific rate and the adjusted rate are subject to random variation (random error) and any expression of the rates must take this variation into account. A measure of the variation is the standard deviation, or the standard error, of a rate. We need the standard deviation in order to use the rates in estimation, for testing hypotheses, or for making other statistical inferences concerning the mortality of a population. With the standard deviation one can assess the degree of confidence that may be placed in the findings and conclusions reached on the basis of these rates. With the standard deviation one can also measure the quality of the vital statistics and, in fact, evaluate the reliability of the rates themselves.

Since a death rate is often determined from the mortality experience of an entire population rather than from a sample, it is sometimes argued that there is no sampling error; and therefore the standard deviation, if it exists, can be disregarded. This point of view, however, is static. Statistically speaking, human life is a random experiment and its outcome, survival or death, is subject to chance. If two people were subjected to the same risk of dying (force of mortality) during a calendar year, one might die during the year and the other survive. If a person was allowed to relive the year
he survived the first time, he might not survive the second time.

Similarly, if a population were allowed to live the same year over again, the total number of deaths occurring during the second time would assume a different value and so, of course, would the corresponding death rate. It is in this sense that a death rate is subject to random variation even though it is based on the total number of deaths and the entire population.

From a theoretical viewpoint, a death rate is an estimate of certain functions of the force of mortality acting upon each individual and may assume different values with correspondingly different probabilities, even if the force of mortality remains constant. Therefore, it is natural and meaningful to study the standard deviation of a rate.

Age-specific death rates, when they are determined from a sample, are subject to sampling variation in addition to random variation. The standard deviation of a death rate assumes different forms, depending upon the sampling unit and sampling procedure used. But generally it consists of two components: one due to sampling, and the other due to experimentation (the chance of surviving the year). The standard deviation of a death rate based on a sample will be discussed in Section 4. At present, we will discuss the standard deviation of death rates subject to random variation only.

REMARK. The terms "standard deviation" (of a death rate) and "standard error" (of a death rate) have the same meaning. They are the square root of the variance (of a death rate), and both are commonly used in statistics and in mortality analysis. To acquaint the reader with both terms, we shall use "standard deviation" and "standard error" alternately in this manual.
The basic concept used in application of statistical inference to death rates is the binomial distribution and the central limit theorem. Consider a sequence of independent trials, each trial having either of two possible outcomes, i.e., "success" or "failure," with the corresponding probabilities remaining the same for all trials. Such trials are called Bernoulli trials. Tossing a coin is a familiar example: each toss of a coin constitutes a trial (a random experiment) with either of two possible outcomes, heads or tails. A person's life over a year is another example with the corresponding outcomes of survival or death during the year. The binomial random variable is the number of "successes" in a number of independent and identical trials, each trial can result either in a "success" or a "failure" and the probability of a "success" is the same for all trials. Thus a binomial random variable is the number of "successes" in a number of Bernoulli trials. The number of heads shown in a number of tosses of a coin is a binomial random variable.

If \( N_1 \) people alive at exact age \( x_i \) are subject to the same probability \( q_i \) of dying in the age interval \( (x_i, x_{i+1}) \), the number of people \( D_i \) dying in the interval is also a binomial random variable. The expected number of deaths, denoted by \( E(D_i) \) is

\[
E(D_i) = N_1 q_i \quad (2.1)
\]

and the variance of \( D_i \) is

\[
\sigma^2_{D_i} = N_1 q_i (1-q_i) \quad (2.2)
\]

The proportion of deaths, or the binomial proportion,

\[
\frac{D_i}{N_i} = q_i \quad (2.3)
\]
is an unbiased estimate of the probability \( q_1 \) in the sense that its expected value is equal to \( q_1 \)

\[
E(\hat{q}_1) = E\left(\frac{D_1}{N_1}\right) = \frac{1}{N_1} E(D_1) = \frac{1}{N_1} N_1 q_1 = q_1.
\]  

(2.4)

The variance of \( \hat{q}_1 \), which may be derived from (2.2), is given by

\[
\sigma_{\hat{q}_1}^2 = \frac{1}{N_1} q_1 (1-q_1).
\]  

(2.5)

When the probability \( q_1 \) is unknown, its estimate \( \hat{q}_1 \) is substituted in (2.5) to give the "sample" variance of \( \hat{q}_1 \),

\[
S_{\hat{q}_1}^2 = \frac{1}{N_1} \hat{q}_1 (1-\hat{q}_1).
\]  

(2.6)

Both the variance in (2.5) and the sample variance in (2.6) are measures of variation associated with the proportion \( \hat{q}_1 \) and play an important role in making inferences concerning the unknown probability \( q_1 \). The fundamental theorem needed in this situation is the central limit theorem. According to the theorem, when \( N_1 \) is sufficiently large, the standardized form of the random variable \( \hat{q}_1 \),

\[
Z = \frac{\hat{q}_1 - q_1}{\sqrt{q_1 (1-q_1)/N_1}}
\]  

(2.7)

has the standard normal distribution with a mean of zero and a variance of one.
Formula (2.7) expresses the deviation of the random variable $\hat{q}_i$ from its expected value $q_i$ in units of the standard deviation $\sigma_i$. Using formula (2.7), one can test a hypothesis concerning the probability $q_i$ or estimate $q_i$ by means of a confidence interval.

Suppose a study of infant mortality in a community suggests a decline in infant deaths. A hypothesis concerning the probability of death in the first year of life, $q_0 = .028$ (or .28 per 1,000), is to be tested against an alternative hypothesis $q_0 < .208$. The statistic used to test the hypothesis is the quantity in (2.7) with the substitution of $q_0 = .028$, or

$$Z = \frac{\hat{q}_0 - .028}{\sqrt{(\frac{.028}{1-.028})/N_0}} \quad (2.8)$$

where $N_0$, the number of newborns in the study and $\hat{q}_0 = D_0/N_0$, the proportion of infant deaths, can be determined from the data observed, and the quantity in (2.8) can be computed. Rejection or acceptance of the hypothesis $q_0 = .028$ is based on the computed value of (2.8). At the 5% level of significance, for example, the hypothesis is rejected if the computed value of $Z$ is less than -1.645, the fifth percentile in the standard normal distribution.

One may also use (2.7) and the normal distribution percentiles to determine confidence intervals for the probability $q_i$. For a .95 confidence coefficient, for example, we use the 2.5 percentile of -1.96 and the 97.5 percentile of +1.96. This means that

$$\Pr\{-1.96 < \frac{\hat{q}_i - q_i}{\sqrt{\frac{q_i(1-q_i)}{N_i}}} < 1.96\} = .95 \quad (2.9)$$

The inequalities inside the braces are approximately equivalent to

$$\hat{q}_i - 1.96 \sigma_i < q_i < \hat{q}_i + 1.96 \sigma_i \quad (2.10)$$
where the sample standard deviation, $S_q$, is the square root of the variance $q_i$ in (2.6). The inequalities in (2.10) provide the fundamental formula for the 95% confidence interval for the probability $q_i$.

3. Probability of Death and the Age-specific Death Rate

The probability of death and the age-specific death rate are two measures of the risk of mortality acting on individuals in the population. While the probability of death is an established concept in the field of statistics, analytic meaning of the age-specific death rate is not fully appreciated. The age-specific death rate either is regarded as an ill-defined statistical quantity, or else it is treated as if it were another name for the probability of death. These misconceptions need be corrected. The age-specific death rate is just as meaningful analytically as the probability. The exact meaning of the age-specific death rate and its relationship with the probability of death have been given in Chapter 2 and will be discussed in more detail in Chapter 5. For easy reference, we state again the estimate of the probability and the age-specific death rate below.

Let $N_i$ be the number of individuals alive at the exact age $x_i$, among them a number $D_i$ dying during the interval $(x_i, x_{i+1})$. Then the estimate of the probability of dying in $(x_i, x_{i+1})$ is given by (cf. equation (2.3)),

$$\hat{q}_i = \frac{D_i}{N_i} \quad .$$

On the other hand, the age-specific death rate, $M_i$, is the ratio of the number of deaths, $D_i$, to the total number of years lived in the interval $(x_i, x_{i+1})$ by the $N_i$ people. In formula

$$M_i = \frac{D_i}{n_i(N_i-D_i)+a_i n_1 D_1} \quad .$$

Solving equations (3.1) and (3.2) yields the basic relationship between $\hat{q}_i$ and $M_i$. 

Here \( n_i = x_{i+1} - x_i \), and \( a_i \) is the average fraction of the age interval \((x_i, x_{i+1})\) lived by individuals dying at any age included in the interval. The fraction \( a_i \) has been computed for a number of countries whose population and mortality data are available; the values of \( a_i \) are given in Appendix V.

For a current population, the age-specific death rates are determined from the vital and population statistics,

\[
M_i = \frac{D_i}{P_i} \tag{3.4}
\]

where \( D_i \) is the number of deaths occurring in age group \((x_i, x_{i+1})\) during a calendar year and \( P_i \) is the corresponding mid-year population. The probability of death \( \hat{q}_i \) is computed from formula (3.3).

To determine the variance of \( \hat{q}_i \), we start with formula (2.6)

\[
S^2_{\hat{q}_i} = \frac{1}{N_i} \hat{q}_i (1 - \hat{q}_i) . \tag{2.6}
\]

Since equation (3.1) implies that

\[
\frac{1}{N_i} = \frac{1}{D_i} \hat{q}_i ,
\]

we have the desired formula for the sample variance of \( \hat{q}_i \):

\[
S^2_{\hat{q}_i} = \frac{1}{D_i} \hat{q}_i^2 (1 - \hat{q}_i) . \tag{3.5}
\]

The exact formula for the variance of the age-specific death rate is difficult to derive. However, since the population size \( P_i \) in (3.4) usually is large, we use Taylor's expansion to establish the following relationship between the variance of \( M_i \) and the variance of \( D_i \):

\[
\hat{q}_i = \frac{n_i M_i}{1 + (1-a_i)n_i M_i} . \tag{3.3}
\]
where the sample variance of $D_1$ is

$$S_{D_1}^2 = N_1 \hat{q}_1 (1-\hat{q}_1) = D_1 (1-\hat{q}_1).$$ (3.7)

Substituting formula (3.7) in (3.6) yields the required formula for the sample variance of the age-specific death rate

$$S_{M_1}^2 = \frac{1}{P_1} M_1 (1-\hat{q}_1).$$ (3.8)

When $\hat{q}_1$ is very small so that $1-\hat{q}_1$ is close to one, formulas (3.5) and (3.8) may be approximated by

$$S_{q_1}^2 = \frac{1}{D_1} \hat{q}_1^2$$ (3.9)

and

$$S_{M_1}^2 = \frac{M_1}{P_1}.$$ (3.10)

respectively.
4. The Death Rate Determined from a Sample

It should be emphasized that although $N_i$ and $P_i$ in the above discussion both refer to the numbers of people in a population, the formulas of sample variances of $q_i$ and $M_i$ in (3.5) and (3.8) hold also when $N_i$ and $P_i$ are the numbers of people in a sample. To verify this, suppose a random sample of $N$ people is taken from an entire population. In the sample there are $N_i$ people of age $x_i$, $D_i$ of whom die during the year, and

$$
\frac{D_i}{N_i} = q_i
$$

is an estimate of the probability $q_i$. We are interested in the sample variance of $q_i$. In formula (4.1) both the numerator and the denominator are random variables; $N_i$ is subject to sampling variation in the sense that the number of people of age $x_i$ included in the sample varies from one sample to another, while $D_i$ is subject to sampling variation and random variation (survival or death during the year). The formula for the variance of the ratio in (4.1) thus can be expressed in terms of the variance of $N_i$ and of $D_i$.

However, the variance of $D_i$ consists of two components: the random component and the sampling component. The derivation of the variance of $\hat{q}_i$ through the variance of $D_i$ is lengthy. To save space, we use the following simpler approach to derive the variance of $\hat{q}_i$ directly.

It is easy to verify that given $N_i$ the conditional expectation and conditional variance of $\hat{q}_i$ are, respectively,

$$
E(\hat{q}_i|N_i) = q_i
$$

and

$$
\sigma_{\hat{q}_i|N_i}^2 = \frac{1}{N_i} q_i(1-q_i).
$$

*/ For simplicity in demonstrating our reasoning, but at the expense of a certain degree of reality, we assume $N_i$ people of exact age $x_i$. 

On the other hand, because of (4.2), the variance of \( \hat{q}_i \) is equal to the expected value of the conditional variance of \( \hat{q}_i \) given \( N_i \),

\[
\sigma^2_{\hat{q}_i} = \mathbb{E}(\sigma^2_{\hat{q}_i} | N_i).
\]  

(4.4)

Substituting (4.3) in (4.4) gives

\[
\sigma^2_{\hat{q}_i} = \mathbb{E}(\frac{1}{N_i} q_i (1-q_i)).
\]  

(4.5)

Using the sample information, we obtain the sample variance of \( \hat{q}_i \)

\[
S^2_{\hat{q}_i} = \frac{1}{N} \hat{q}_i (1-\hat{q}_i) = \frac{1}{D_i} \sigma^2_1 (1-q_1),
\]  

(4.6)

since \( N_i \) is given in (4.1). This shows that although \( \hat{q}_i \) in (4.1) is computed from a sample, its sample variance has the same expression as the variance of \( \hat{q}_i \) based on a total population. It is easy to justify now that the variance of the age-specific death rate in (3.8) holds true also when the death rate is computed on the basis of a sample.
5. Age-Adjusted Death Rates and Mortality Indices

In Chapter 2 several methods of adjustment of age-specific death rates were presented. Although each method was developed on the basis of a specific philosophic argument and designed to serve a definite purpose, they all assume a general form of a weighted mean of the age-specific death rates. These methods of adjustment are reproduced in Table 1 for easy reference.

With the exception of the indirect method of adjustment, the weights add to unity. The sum of the weights in the indirect method can be greater or less than unity, depending upon the difference between community and standard populations in age composition. For this reason, the indirect method is not strictly comparable with any other adjusted rate, and neither is its standard error.

The inclusion of the crude death rate in the list of adjusted rates is of significance. Since it is usually expressed as the ratio of all deaths to the total midyear population, the crude death rate is occasionally treated as a binomial proportion, which leads to an incorrect formula for the standard deviation. Individuals differing in age and sex obviously do not have the same probability of dying, and the notion of an average probability is incomprehensible; therefore, a direct application of the binomial theory is inappropriate. If, however, it is visualized as the weighted mean of specific death rates, with the actual population size employed as weights, then the crude death rate is perhaps the most meaningful measure of mortality for a single community. This way of viewing the crude rate is also essential in the derivation of its standard deviation.

In all the adjusted rates, the choice of weights applied to specific rates is based on: (1) the proportion of those in a specific age group
to the total population, i.e., population proportion, and (2) the relative
interval length of a specific age group. For the crude rate, the weights
used are the community population proportions in specific age groups
\( P_{ui}/P_u \); for the direct method of adjustment, the standard population
proportions in specific age groups \( P_{si}/P_s \); for the comparative mortality
rate, the average of the two population proportions; for the life table
dearth rate, the life table population proportions for specific age groups
\( L_1/T_0 \); and for the equivalent average death rate, the relative interval
lengths of the age groups \( n_i/\Sigma n_i \). The weights used in the indirect method
of adjustment are functions of the age-specific rate for the standard
population, community population proportions, and standard population proportions.

The methods of adjustment listed in Table 1 also include two indices,
the relative mortality index, the mortality index, and the standardized mortality ratio.
As seen from the second panel of Table 1, the two indices are weighted means of the
ratios of a community's specific death rates to the corresponding specific rates for
the standard population. The difference is that the relative mortality
index uses the population proportions of the community for specific age
groups as weights, while the mortality index uses the relative lengths of the
age intervals. In the derivation of their standard deviations, however,
we shall consider them as linear functions of age-specific death rates of
a community with coefficients as listed in the weight column.
Table 1. Formulas and weights used to compute the crude death rate, age-adjusted rates and mortality indices

<table>
<thead>
<tr>
<th>Title</th>
<th>Formula</th>
<th>Weight ($w_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude death rate (C.D.R.)</td>
<td>$\sum \frac{P_{ui}M_{ui}}{P_u}$</td>
<td>$\frac{P_{ui}}{P_u}$</td>
</tr>
<tr>
<td>Direct method of adjustment (D.M.D.R.)</td>
<td>$\frac{\sum P_{si}M_{ui}}{P_s}$</td>
<td>$\frac{P_{si}}{P_s}$</td>
</tr>
<tr>
<td>Comparative mortality rate (C.M.R.)</td>
<td>$\frac{1}{2} \left( \frac{P_{ui}}{P_u} + \frac{P_{si}}{P_s} \right)M_{ui}$</td>
<td>$\frac{1}{2} \left( \frac{P_{ui}}{P_u} + \frac{P_{si}}{P_s} \right)$</td>
</tr>
<tr>
<td>Indirect method of adjustment (I.M.D.R.)</td>
<td>$\frac{D/P_s(D/u)}{P_u} \sum \frac{P_{ui}M_{si}}{P_u}$</td>
<td>$\frac{D/P_s}{P_u} \sum \frac{P_{ui}M_{si}}{P_u}$</td>
</tr>
<tr>
<td>Life table death rate (L.T.D.R.)</td>
<td>$\frac{\sum L_{ui}}{\sum L_i}$</td>
<td>$\frac{L_i}{\sum L_i}$</td>
</tr>
<tr>
<td>Equivalent average death rate (E.A.D.R.)</td>
<td>$\frac{\sum n_iM_{ui}}{\sum n_i}$</td>
<td>$\frac{n_i}{\sum n_i}$</td>
</tr>
<tr>
<td>Relative mortality index (R.M.I.)</td>
<td>$\frac{\sum P_{ui}M_{ui}}{P_u M_{si}}$</td>
<td>$\frac{P_{ui}}{P_u M_{si}}$</td>
</tr>
<tr>
<td>Mortality index (.M.I.)</td>
<td>$\frac{\sum n_i M_{ui}}{\sum n_i}$</td>
<td>$\frac{n_i}{(\sum n_i) M_{si}}$</td>
</tr>
<tr>
<td>Standardized mortality ratio (S.M.R.)</td>
<td>$\sum \frac{P_{ui}M_{ui}}{EP_{ui}M_{si}}$</td>
<td>$\frac{P_{ui}}{EP_{ui}M_{si}}$</td>
</tr>
</tbody>
</table>
6. Sample Variance of the Age-Adjusted Death Rate

To derive the formula for the sample variance of an adjusted death rate, it is first essential to identify the random variables involved. Clearly, $M_{ui}$, the age-specific death rates for a community, are random variables while $n_i$, the interval length for an age group, is a constant. Community and standard population proportions for specific age groups will not be treated as random variables for the reason that the random event under study is death, not population. The age-specific death rates of the standard population are random variables, just as are the community age-specific death rates. However, since adjusted death rates are derived for the purpose of testing hypotheses concerning the mortality experience of communities, only that part of the random variation associated with the communities in question should be taken into consideration. In other words, random variations attributable to the age-specific rates for the standard population should not be included in the variance of the adjusted rates. Life table population proportions, on the other hand, are derived from the age-specific death rates for a community; therefore, they should be treated as random variables. To summarize, we shall consider only the community age-specific death rates and the life table population proportions for specific age groups as random variables in the derivation of the sample variance.

With this understanding, and making an exception of the life table death rate, we shall write adjusted rates and mortality indices as linear functions of the basic random variables, the age-specific death rates of a community. The general formula for an adjusted death rate or mortality index $R$ takes the form

$$R = \sum_{i} w_i M_{ui}$$

(6.1)

with the coefficient $w_i$ as given in Table 1. The general rules for the
variance of a linear function of random variables may now be applied and the variance of the adjusted rate, \( R \), may be expressed as follows:

\[
S_R^2 = \sum_i w_i^2 S_{M_i}^2 + \sum_{i \neq j} w_i w_j S_{M_i M_j},
\]

where \( S_{M_i}^2 \) is the sample variance of the age-specific death rate for age group \( (x_i, x_{i+1}) \) in the community \( u_i \), and \( S_{M_i M_j} \) is the sample covariance between the age-specific death rates, \( M_i \) and \( M_j \).

The age-specific death rate, \( M_{ui} \), is a function of the corresponding estimated probability of death, \( q_i \); and the covariance between death rates is also a function of the covariance between the two corresponding estimated probabilities. It has been proved [cf. Section 5, Appendix II] that the estimated probabilities for two nonoverlapping age intervals have a zero covariance. Thus, two death rates will also have a zero covariance. It follows that all the covariances in formula (6.2) will vanish, and the formula for the sample variance of \( R \) becomes

\[
S_R^2 = \sum_i w_i^2 S_{M_i}^2.
\]

Using (3.8) for the variance of \( M_{ui} \), we have

\[
S_R^2 = \sum_i w_i^2 \frac{M_{ui}}{P_{ui}} (1 - \hat{q}_{ui}),
\]

or using the approximate formula, (3.10), we have

\[
S_R^2 = \sum_i w_i^2 \frac{M_{ui}}{P_{ui}}.
\]
7. Computation of the Sample Variance of the Direct Method Age-Adjusted Death Rate

The computation of the sample variance of the age-specific death rate is the common essential part to all the methods of adjustment except for the life table death rate. Therefore, it is sufficient to use only the direct method of adjustment (D.M.D.R.) as an example. The formula for the sample variance of D.M.D.R. is obtained from (6.4) with \( w_i = \frac{p_{si}}{P_s} \):

\[
S^2 = \sum_{i} \left[ \frac{p_{si}}{P_s} \right] \frac{M_{ui}}{p_{ui}} (1 - q_{ui})
\]

(7.1)

For this illustration, we use the death rates of the total California population of 1970, and the United States 1970 population as the standard population. The steps involved in the computation are shown in Table 2.

The age group 85 and over presents a problem which needs special treatment. Because it is an open-ended group, the interval length is not determinable. The average number of years, \( a_{85}^{n_{85}} \), lived by individuals may be estimated by the reciprocal of the central death rate,

\[
a_{85}^{n_{85}} = \frac{1}{M_{85}}.
\]

(7.2)

Justification of (7.2) is given in Appendix II (cf., equation (8)) on life table construction. Equation (7.2) implies

\[
1 - a_{85}^{n_{85}} M_{85} = 0,
\]

which means that the sample variance of \( M_{85} \), as given in (3.8), is equal to zero. Intuitively, the zero variance can be justified as follows: Each individual alive at age 85 has a future life time of, say, \( y \) years. The sample variance of \( y \) is the mean-square deviation of each \( y \) from the sample mean. If in a group of individuals alive at age 85, the only information
available is that each \( y \) assumes an average of \( a_{85}n_{85} \) years, then the
development of each \( y \) from the sample mean is zero. The sample variance is
also zero. This implies that the sample variance of \( M_{85} \) is zero. It may
be noted also that \( \hat{a}_{85} = 1 \) so that the variance in (3.8) is equal to zero.

For each of the remaining age groups \( (x_i, x_{i+1}) \) we compute the sample
variance of the death rate \( M_{ui} \) [cf., equation (3.8)]

\[
S^2_{M_{ui}} = \frac{M_{ui}}{P_{ui}} (1 - \hat{q}_{ui})
\]  

(7.3)
as shown in column (7) in Table 2 and the corresponding weight squared
\( (P_{si}/P_s)^2 \) in column (8), and find the product

\[
\left( \frac{P_{si}}{P_s} \right)^2 S^2_{M_{ui}} = \left( \frac{P_{si}}{P_s} \right)^2 \frac{M_{ui}}{P_{ui}} (1 - \hat{q}_{ui}).
\]  

(7.4)
Adding the products in (7.4) over all age groups, we obtain the sample
variance of \( R \) in formula (7.1).

For the California 1970 population the age-adjusted death rate is

\[
R = \sum \frac{P_{si}}{P_s} M_{ui}
\]

\[
= \frac{1,787,768.98}{203,211,926} = .0087976
\]
The computation in Table 2 shows that the sample variance is

\[
S^2_R = 340.631 \times 10^{-12}
\]
and the standard deviation is

\[
S_R = \sqrt{340.631 \times 10^{-12}} = 18.456 \times 10^{-6}
\]
In comparison, the standard deviation \( S_R \) is much smaller than the age-adjusted
death rate. Formally, the magnitude of a standard deviation is measured by
the coefficient of variation, which is defined as the ratio
In this case

\[ \text{Coeff. of variation of } R = \frac{S_R}{R} \]  

\[ = \frac{.018456}{8.7976} \]

\[ = .0020979 = .21 \text{ percent} \]

The small magnitude of the coefficient of variance is mainly due to the large population sizes \( P_{ul} \).

The age-adjusted death rate of 8.7976 per 1000 for the California 1970 population may be compared with the total United States population, 1970, death rate, 9.453 per 1000, since both are based on the same population distribution. Because of the small standard deviation (\( S_R = .018456 \) per 1000), we conclude that in 1970 the California population had a significantly lower mortality than the United States as a whole.
Table 2. Computation of sample standard error of the age-adjusted death rate for total population, California, 1970.

(Adjustment made by the direct method. The standard population used is the total population of the United States, enumerated as of April 1, 1970)

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Length of Interval</th>
<th>Mid-year Population in Interval ($x_i$ to $x_{i+1}$)</th>
<th>Death Rate &amp; Probability of Dying in Interval</th>
<th>Sample Variance of Age Specific Death Rate $M_{ui}(1-q_{ui})$</th>
<th>Square of Standard Proportion (U.S., 1970) $10^{9}(P_{si}/P_{s})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$ to $x_{i+1}$</td>
<td>$n_i$</td>
<td>$p_{ui}$</td>
<td>$M_{ui}$</td>
<td>$a_{ui}$</td>
<td>$q_{ui}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>1</td>
<td>340483</td>
<td>.018309</td>
<td>.09</td>
<td>.018015</td>
<td>52805.15</td>
<td>29415.50</td>
</tr>
<tr>
<td>1-5</td>
<td>4</td>
<td>1302198</td>
<td>.000806</td>
<td>.41</td>
<td>.00322</td>
<td>616.96</td>
<td>452458.67</td>
</tr>
<tr>
<td>5-10</td>
<td>5</td>
<td>1918117</td>
<td>.000377</td>
<td>.44</td>
<td>.00188</td>
<td>196.18</td>
<td>964404.79</td>
</tr>
<tr>
<td>10-15</td>
<td>5</td>
<td>1963681</td>
<td>.000374</td>
<td>.54</td>
<td>.00187</td>
<td>190.10</td>
<td>1046618.41</td>
</tr>
<tr>
<td>15-20</td>
<td>5</td>
<td>1817379</td>
<td>.001130</td>
<td>.59</td>
<td>.00564</td>
<td>618.27</td>
<td>880681.46</td>
</tr>
<tr>
<td>20-25</td>
<td>5</td>
<td>1740966</td>
<td>.001552</td>
<td>.49</td>
<td>.00773</td>
<td>884.57</td>
<td>649012.63</td>
</tr>
<tr>
<td>25-30</td>
<td>5</td>
<td>1457614</td>
<td>.001421</td>
<td>.51</td>
<td>.00708</td>
<td>967.98</td>
<td>439832.81</td>
</tr>
<tr>
<td>30-35</td>
<td>5</td>
<td>1219389</td>
<td>.001611</td>
<td>.52</td>
<td>.00802</td>
<td>1310.56</td>
<td>316393.25</td>
</tr>
<tr>
<td>35-40</td>
<td>5</td>
<td>1149999</td>
<td>.002250</td>
<td>.53</td>
<td>.01119</td>
<td>1934.63</td>
<td>298733.21</td>
</tr>
<tr>
<td>40-45</td>
<td>5</td>
<td>1208550</td>
<td>.003404</td>
<td>.54</td>
<td>.01689</td>
<td>2769.03</td>
<td>347603.72</td>
</tr>
<tr>
<td>45-50</td>
<td>5</td>
<td>1245903</td>
<td>.005395</td>
<td>.53</td>
<td>.02654</td>
<td>4214.84</td>
<td>355480.49</td>
</tr>
<tr>
<td>50-55</td>
<td>3</td>
<td>1083852</td>
<td>.008256</td>
<td>.53</td>
<td>.04049</td>
<td>7308.85</td>
<td>298580.84</td>
</tr>
<tr>
<td>55-60</td>
<td>5</td>
<td>933244</td>
<td>.012796</td>
<td>.52</td>
<td>.06207</td>
<td>12860.25</td>
<td>240855.01</td>
</tr>
<tr>
<td>60-65</td>
<td>5</td>
<td>770770</td>
<td>.018565</td>
<td>.52</td>
<td>.08886</td>
<td>21945.99</td>
<td>179800.97</td>
</tr>
<tr>
<td>65-70</td>
<td>5</td>
<td>620805</td>
<td>.027526</td>
<td>.51</td>
<td>.12893</td>
<td>38622.55</td>
<td>118374.42</td>
</tr>
<tr>
<td>70-75</td>
<td>5</td>
<td>484431</td>
<td>.039529</td>
<td>.52</td>
<td>.18052</td>
<td>66868.60</td>
<td>71764.70</td>
</tr>
<tr>
<td>75-80</td>
<td>5</td>
<td>342097</td>
<td>.062336</td>
<td>.51</td>
<td>.27039</td>
<td>132947.58</td>
<td>35611.87</td>
</tr>
<tr>
<td>80-85</td>
<td>5</td>
<td>210953</td>
<td>.095419</td>
<td>.50</td>
<td>.38521</td>
<td>278083.97</td>
<td>12636.07</td>
</tr>
<tr>
<td>85+</td>
<td>-</td>
<td>142691</td>
<td>.157564</td>
<td>-</td>
<td>1.00000</td>
<td>0.00</td>
<td>5528.07</td>
</tr>
</tbody>
</table>
8. Sample Variance of the Life Table Death Rate

The life-table death rate is a special case in that the weights \( \frac{L_x}{\sum L_x} \), as functions of the age-specific death rates, are themselves random variables and are correlated not only with each other, but with the specific death rates as well. Obviously, a derivation of the sample variance of the life-table death rate based on the approach presented in the previous sections will involve a series of complicated and difficult computations.

The derivation can be simplified by making use of the inverse relationship between \( R \), the life-table death rate, and \( \hat{e}_0 \), the observed expectation of life at birth:

\[
R = \frac{\sum L_x M_x u_x}{\sum L_x} = \frac{\sum d_x}{\sum L_x} = \frac{\hat{e}_0}{\hat{e}_0} = \frac{1}{\hat{e}_0}.
\]  

(8.1)

Employing the general rule on the variance of the inverse of a random variable, we have

\[
\sigma_R^2 = \frac{1}{\hat{e}_0^4} \sigma_{\hat{e}_0}^2.
\]

(8.2)

Here the sample variance of \( \hat{e}_0 \), which may be found in Chapter 4, is

\[
\sigma_{\hat{e}_0}^2 = \sum_{x \geq 0} \sigma_{\hat{e}_0 x}^2 \frac{[1-a_x)n_x + \hat{e}_x + n_x]^2}{q_x \cdot \hat{e}_0}.
\]

(8.3)

Substituting (8.3) in (8.2) gives the required formula

\[
\sigma_{\hat{e}_0}^2 = \frac{1}{\hat{e}_0^4} \sum_{x \geq 0} \sigma_{\hat{e}_0 x}^2 \frac{[1-a_x)n_x + \hat{e}_x + n_x]^2}{q_x \cdot \hat{e}_0}.
\]

(8.4)

where

\[
\sigma_{\hat{e}_0 x} = \frac{\hat{q}_x^2 [1-\hat{q}_x]}{d_x}
\]

as given in Section 3.
CHAPTER 4
THE LIFE TABLE AND ITS CONSTRUCTION

An Historical Note

Long before the development of modern probability and statistics, men were concerned with the length of life and constructed tables to measure longevity. Particular interest has been expressed to the longevity of famous persons or to individuals who were reported to have died at an extreme old age. A crude table, credited to the Roman Praetorian Praefect Ulpianus, was constructed in the middle of the third century A.D., and indicates an expectation of life of thirty years. But since its purpose was to serve as a basis for determining annuity grants, it is unlikely that it reflects mortality in the general population. Nevertheless, it continued in official use in northern Italy until the end of the eighteenth century. John Graunt's Bills of Mortality, published in 1662, and Edmund Halley's famous table for the city of Breslau, published in 1693, mark the beginning of modern life tables. In Bills of Mortality, Graunt introduced the proportion surviving to various ages, while Halley's table already contained most of the columns in use today. Rough calculation of the average length of life from Graunt's data for seventeenth century London gives a figure of 18.2 years, whereas Halley's estimate for Breslau near the end of the century was 33.5 years. During the next hundred years several life tables were constructed, including the French tables of Deparcieux (1746), of Buffon (1749), of Mourgue and Duvillard (both published in the 1790's), the Northampton table of Richard Price (1763), and in the United States Wigglesworth's table for Massachusetts and New Hampshire (1793). The first official English life table was published in 1843 during William Farr's term as Compiler of Abstracts in the General
Records Office. Several countries in Continental Europe have established series of life tables dating back almost two centuries. Sweden, for example, began a series of life tables in 1755, Netherlands in 1816, France in 1817, Norway in 1821, Germany in 1871 and Switzerland in 1876. Reliable mortality statistics for the construction of United States life tables did not become available until 1900; from there J. W. Glover, of the Bureau of the Census, determined that the expectation of life at birth was 46.07 years for males and 49.42 for females.

1. Introduction

The life table is largely a product of actuarial science, but its application is not limited to the computation of insurance premiums. Recent advances in theoretical statistics and stochastic processes have made it possible to study the length of life from a purely statistical point of view, making the life table a valuable analytical tool for demographers, epidemiologists, physicians, and research workers in other areas of public health.

There are two forms of the life table in general: the cohort (or generation) life table and the current life table. In its strictest form, a cohort life table records the actual mortality experience of a particular group of individuals (the cohort) from birth to the death of the last member of the group. The difficulties involved in constructing a cohort life table for a human population are apparent. Statistics covering a period of 100 years are available for only a few populations and even those are likely to be less reliable than current statistics. Individuals in a given cohort may have emigrated or died unrecorded, and the life expectancy of a group of people already dead is of little more than historical interest. However, cohort life tables do have practical applications in studying animal populations and have even been extended to access the durability of inanimate objects such as engines, electric light bulbs, etc. Modified or adapted cohort tables
have been useful in epidemiological, sociological, and medical and para-
medical studies with human subjects. Extensive use of life table methods
has been made in the analysis of chance and duration of patient-survival
in studies of treatment effectiveness. These will be discussed in more detail
with some examples in Chapter 9.

The current life table, as the name implies, gives a cross-section view
of the mortality and survival experience of all ages in a population during
one short period of time, for example, the California population of
1970. It is dependent entirely on the age-specific death rates prevailing
in the year for which it is constructed. Such tables project the life span
of each individual in a hypothetical cohort on the basis of the actual death
rates in a given population. When we speak of the life expectancy of an
infant born in a current year, for example, we mean the life expectancy
that would be obtained if he were subjected throughout his life to the
same age-specific mortalities prevailing in the current year. The current
life table is then a fictitious pattern reflecting the mortality experience
of a real population during a calendar year. However, it is the most effective
means of summarizing mortality and survival experience of a population, and
is a sound basis for making statistical inference about the population under
study. The reader can no doubt confirm from his own experience that the
current life table is a standard and useful tool for comparing international
mortality data, and for assessing mortality trends on the national level.

A current life table may be based on the deaths occurring over three,
instead of one, calendar years; e.g., years 1969, 1970, 1971. For each age
group the average number of deaths per year is then divided by the corres-
ponding population size of the middle of the three years (1970, in this
example) to obtain the age-specific death rate. Usually, the middle year
is a census year, so that population figures are available and more accurate.
The advantage of such a table is to reduce the possible abnormalities in mortality pattern which may exist in a single calendar year.

Data for constructing life tables are sometimes refined by graduation or other methods for smoothing or reducing the effect of extreme values. Techniques for refinement of life table data were developed by actuarial scientists. While refinement of data has its merit in smoothing data, it is difficult to make proper statistical inference of life table functions which is based on such information.

This chapter will describe a general form of the life table with interpretations of its various functions and present a method of constructing a current life table. Theoretical aspects of life table functions will be discussed in detail in Appendix II.

Cohort and current life tables may be either complete or abridged. In a complete life table the functions are computed for each year of life; an abridged life table differs only in that it deals with age intervals greater than one year, except possibly the first year of the first five years of life. A typical set of intervals is 0-1, 1-5, 5-10, 10-15, etc.
2. Description of the Life Table

Cohort and current life tables are identical in appearance but different in construction. The following discussion refers to the complete current life table. The function of each column is defined and its relation to the other columns explained; conventional symbols have been modified for the sake of simplicity. The complete current life table for the California total population in 1970, presented in Table 2, will serve as an example.

Column 1. Age interval, \((x, x+1)\) -- As with the cohort table, each interval in this column is defined by the two exact ages stated except for the final age interval, which is open-ended such as 85 and over. The starting point for the final age interval is denoted by \(w\).

Column 2. Proportion (of those alive at age \(x\)) dying in interval \((x, x+1)\), \(q_x\) -- Each \(q_x\) is an estimate of the probability that an individual alive at the exact age \(x\) will die during the interval. These proportions are the basic quantities from which figures in other columns of the table are computed. They are derived from the corresponding age-specific death rates of the current population, using formulas that will be explained in the next section. To avoid decimals, the proportions are sometimes expressed as the number of deaths per 1,000 population, and the column is headed, "1000 \(q_x\)."

Column 3. Number alive at age \(x\), \(l_x\) -- The first number in this column, \(l_0\), is an arbitrary figure called the "radix," while each successive figure represents the number of survivors at the exact age \(x\). Thus the figures in this column have meaning only in conjunction with the radix \(l_0\), and do not describe any observed population. The radix is usually assigned a convenient number, such as 100,000. Table 2 shows that \(l_2\) or 98,088 of every 100,000 persons born alive will survive to the second birthday, provided they are subject to the same mortality experience as that of the 1970 California population.
Column 4. Number dying in interval \((x, x+1)\), \(d_x\) -- The figures in this column are the product of \(l_x\) and \(\hat{q}_x\) and thus also depend upon the radix \(l_0\). Again using the 1970 California experience, we see that out of \(l_0 = 100,000\) born alive, \(d_0 = 1801\) will die in the first year of life. But the number 1801 is meaningless by itself, and is certainly not the number of infant deaths occurring in California in 1970. For each age interval \((x, x+1)\), \(d_x\) is merely the number of life table deaths.

The figures in the columns \(l_x\) and \(d_x\) are computed from the values of \(\hat{q}_0, \hat{q}_1, \ldots, \hat{q}_x\) and the radix \(l_0\) by using the relations

\[
d_x = l_x \hat{q}_x, \quad x=0,1,\ldots,w, \tag{2.1}
\]

and

\[
l_{x+1} = l_x - d_x, \quad x=0,1,\ldots,w-1. \tag{2.2}
\]

Starting with the first age interval, we use equation (2.1) for \(x=0\) to obtain the number \(d_0\) dying in the interval \((0,1)\) and equation (2.2) for \(x=0\) to obtain the number \(l_1\) who survive to the end of the interval. With \(l_1\) persons alive at the exact age 1, we again use the relations (2.1) and (2.2) for \(x=1\) to obtain the corresponding figures for the second interval. By repeated applications of (2.1) and (2.2) we compute all the figures in columns 3 and 4.

Column 5. Fraction of last year of life for age \(x\), \(a'\) -- Each of the \(d_x\) people who die during the interval \((x, x+1)\) has lived \(x\) complete years plus some fraction of the year \((x, x+1)\). The average of these fractions, denoted by \(a'\), plays an important role in the construction of life tables, and in the theoretical studies of life table functions as presented in Appendix II. This will be explained more fully in the next section.
Column 6. Number of years lived by the total cohort in interval \((x,x+1)\), \(L_x\) -- Each member of the cohort who survives the year \((x,x+1)\) contributes one year to \(L_x\), while each member who dies during the year \((x,x+1)\) contributes, on the average, a fraction \(a_x\) of a year, so that

\[
L_x = (l_x - d_x) + a_x d_x, \quad x=0,1,\ldots,w-1,
\]

where the first term on the right side is the number of years lived in the interval \((x,x+1)\) by the \((l_x - d_x)\) survivors, and the last term is the number of years lived in \((x,x+1)\) by the \(d_x\) persons who died during the interval. When \(a_x = 1/2\) (which is usually the case for ages greater than 5), then

\[
L_x = \frac{2}{x} - \frac{1}{2x}, \quad x=0,1,\ldots,w.
\]

The similarity of \(L_x\) to the concept of "person years" may be recognized by the reader.

Column 7. Total number of years lived beyond age \(x\), \(T_x\) -- This total is essential for computation of the life expectancy. It is equal to the sum of the number of years lived in each age interval beginning with age \(x\), or

\[
T_x = L_x + L_{x+1} + \ldots + L_w, \quad x=0,1,\ldots,w,
\]

with an obvious relationship

\[
T_x = L_x + T_{x+1}.
\]

Column 8. Expectation of life at age \(x\), \(\hat{e}_x\) -- This is number of years, on the average, yet to be lived by a person now aged \(x\). Since the total number of years of life remaining to the \(L_x\) individuals is \(T_x\),

\[
\hat{e}_x = \frac{T_x}{L_x}, \quad x=0,1,\ldots,w.
\]
Each $\hat{\epsilon}_x$ summarizes the mortality experience of persons beyond age $x$ in the current population under consideration, making this column the most important in the life table. Further, this is the only column in the table other than $\hat{a}_x$ and $a'_x$ that is meaningful without reference to the radix $\ell_0$. As a rule, the expectation of life $\hat{\epsilon}_x$ decreases as the age $x$ increases, with the single exception of the first year of life where the reverse is true due to the high mortality during the first year. In the 1970 California population, for example, the expectation of life at birth is $\hat{\epsilon}_0 = 71.90$ years whereas at age one $\hat{\epsilon}_1 = 72.22$. The symbol $\hat{\epsilon}_x$, denoting the observed expectation of life, is computed from the actual mortality data and is an estimate of $\epsilon_x$, the true unknown expectation of life at age $x$.1/

Remark 1: Useful quantities which are not listed in the conventional life table are

\begin{equation}
\hat{\epsilon}_x = 1 - \hat{a}_x ,
\end{equation}

the proportion of survivors over the age interval $(x, x+1)$, and

\begin{equation}
\hat{p}_{xy} = \hat{p}_x \hat{p}_{x+1} \cdots \hat{p}_{y-1} \ell_x^{-y} ,
\end{equation}

the proportion of those living at age $x$ who will survive to age $y$. When $x=0$, $\hat{p}_{0y}$ becomes the proportion of the total born alive who survive to age $y$; clearly

\begin{equation}
\hat{p}_{0y} = \ell_y / \ell_0 .
\end{equation}
3. Construction of the Complete Current Life Table

In the construction of current life tables, we are mainly concerned with the computations of $q_x$, the proportion dying in the age interval $(x, x+1)$, and $l_x$, the number of years lived by the radix $l_0$ in the interval $(x, x+1)$.

An important element in complete life table construction as described in this section is the fraction of the last year of life lived by those who die at each age; for example, a man who dies at age 30 has lived 30 complete years plus a fraction of the 31st year. The average value of this fraction is denoted by $a'_x$, where $x$ refers to the age at the last birthday. It might reasonably be expected that the average value of this fraction is equal to one half on the assumption that there are as many deaths at 30 years plus one month as at 30 years plus two months, and at each month thereafter through the 11th; or, in other words, on the assumption that deaths occur uniformly throughout each year of age. Extensive studies of the fraction have been made using the 1960 California mortality data (Chiang, et al [1961]) collected by the State of California Department of Public Health and the 1963 U.S. data collected by the National Vital Statistics Division of the National Center for Health Statistics, respectively. The results obtained so far show that from age 5 on the fractions $a'_x$ are invariant with respect to race, sex, and age, and that the assumed value of .5 is then valid. But a much smaller value has been observed for the first year because of the large proportion of infant deaths occurring in the first weeks of life. A brief description of the analysis regarding $a'_x$ using the California data is given in Section 5.
To return to the computation of \( \hat{q}_x \), readers familiar with vital statistics terminology will recognize the resemblance between \( \hat{q}_x \) and \( M_x \), the age specific death rate. The essential step in constructing a complete life table for a current population is to establish a relationship between \( \hat{q}_x \) and \( M_x \) so that the probability of dying \( q_x \) can be computed from the death rate \( M_x \) for each age \( x \). These two quantities can both be expressed in terms of the number of observed deaths of age \( x \) (\( D_x \)) that occur during the calendar year and the corresponding midyear (calendar year) population (\( P_x \)). Let \( N_x \) be the number of people alive at the exact age \( x \), among whom \( D_x \) deaths occur in \((x,x+1)\). Then, by definition, the proportion died is given by

\[
\hat{q}_x = \frac{D_x}{N_x} \quad .
\]

(3.1)

The age specific death rate, \( M_x \), is the ratio of the number of deaths (\( D_x \)) to the total number of years lived by the \( N_x \) people during the interval \((x,x+1)\). This total number is composed of \((N_x - D_x)\) years lived by the survivors and the number of years by those dying during the year. Let \( a'_x \) be the fraction of the year \((x,x+1)\) lived by a person who dies during the year; then \( D_x \) people as a group will live \( a'_x D_x \) years. Hence the total number of years lived in \((x,x+1)\) is \((N_x - D_x) + a'_x D_x\) and the formula

\[
M_x = \frac{D_x}{(N_x - D_x) + a'_x D_x} \quad .
\]

(3.2)

When the denominator is estimated with the corresponding mid-year population \( P_x \),

\[
(N_x - D_x) + a'_x D_x = P_x \quad ,
\]

(3.3)
we have the familiar formula

$$M_x = \frac{D_x}{P_x}$$  \hspace{1cm} (3.4)

Now, $N_x$, which was introduced to establish a relationship between $\hat{q}_x$ and $M_x$, is nevertheless an unknown quantity. By eliminating $N_x$ from (3.1) and (3.2) and using (3.4), we obtain the desired relationship. Formally, we derive $N_x$ from (3.3):

$$N_x = P_x + D_x - a'_x D_x$$  \hspace{1cm} (3.5)

or

$$N_x = P_x + (1-a'_x)D_x$$

and substituting (3.5) in (3.1) to obtain

$$\hat{q}_x = \frac{D_x}{P_x + (1-a'_x)D_x}$$  \hspace{1cm} (3.6)

Since the age-specific death rate is usually available, we may divide both the numerator and denominator of (3.6) by $P_x$ to obtain the basic formula

$$\hat{q}_x = \frac{M_x}{1 + (1-a'_x)M_x}$$  \hspace{1cm} (3.7)

As it was noted earlier, the fraction $a'_x$ is subject to little variation.

The California data suggest values: $a'_0 = .09$, $a'_1 = .43$, $a'_2 = .45$, $a'_3 = .47$, $a'_4 = .49$, $a'_x = .50$ for $x \geq 5$. Formula (3.7) is fundamental in the construction of complete life tables by the present method and was suggested in Chiang [1960b], [1961].

To illustrate, let us consider the 1970 California population as shown in Table 1. For the first year of life we have $P_0 = 340,483$ in Column 2 and $D_0 = 6,234$ in Column 3. Thus, the age-specific rate for $x=0$ is
\[ M_0 = \frac{D_0}{F_0} = \frac{6.234}{340,483} = 0.018309 \] or 18.309 per 1000.

The average fraction of the year lived by an infant who dies in his first year of life is \( a_0' = 0.09 \). Therefore, the estimate of the probability of dying is computed from (3.1):

\[ \hat{q}_0 = \frac{0.018309}{1 + (1 - 0.09) \cdot 0.018309} = 0.01801. \]

When all the values of \( \hat{q}_x \) have been computed and \( l_0 \) has been selected, \( d_x \) and \( l_x \) for successive values of \( x \) are determined from equations (2.1) and (2.2) as shown in Table 2. For the 1970 California population we determine first the number of life table infant deaths with \( l_0 = 100,000 \),

\[ d_0 = l_0 \hat{q}_0 = 100,000 \cdot 0.01801 = 1801, \]

and the life table survivors at age one,

\[ l_1 = l_0 - d_0 = 100,000 - 1801 = 98199. \]

The formula for the number \( L_x \) of years lived in the age interval \((x, x+1)\) is derived also with the aid of \( a_x' \), the fraction of the last year of life as given in Section 2:

\[ L_x = (l_x - d_x) + a_x'd_x, \quad x = 0, 1, \ldots. \]

To take again the example of the first year of life, \( a_0' = 0.09 \) and

\[ L_0 = 98199 + 0.09 \times 1801 = 98361. \]

Remark 3: The ratio \( d_x/L_x \) is shown as the life-table death rate for age \( x \). Since a life table is entirely based on the age-specific death rates of a current population, the death rates computed from the life table should be identical to the corresponding rates of the current population; symbolically
To prove equation (3.8), we substitute (2.3) in the left side of (3.8) and divide the resulting expression by \( L_x \) to obtain

\[
\frac{d_x}{L_x} = \frac{D_x}{P_x} = \frac{\hat{q}_x}{1 - (1-a_x')\hat{q}_x} \quad (3.9)
\]

Substituting (3.6) for \( \hat{q}_x \) in (3.9) and simplifying the resulting expression give \( D_x/P_x \), proving the assertion (3.8).

The final age interval in a life table is a half-open interval, such as age 85 and over. The values of \( D_w, P_w, M_w, l_w, d_w \), and \( T_w \) all refer to the open interval age \( w \) and over, and \( \hat{q}_w = 1 \) (since there can be no survivors). The length of the interval is infinite and the necessary information for determining the average number of years lived by an individual beyond age \( w \) is unavailable. We must therefore use an approach other than equation (2.3) to determine \( L_w \). Writing the first equation in (3.8) for \( x = w \), we have

\[
L_w = \frac{d_w}{M_w} \quad (3.10)
\]

Since each one of the \( l_w \) people alive at \( w \) will eventually die, \( l_w = d_w \), and from (3.10) we have the required formula

\[
L_w = \frac{l_w}{M_w} \quad (3.11)
\]

where \( l_w \), survivors to age \( w \), is computed from the preceding interval and \( M_w \) is the mortality rate for age interval \( w \) and over. The quantities \( T_w \) and \( \hat{e}_w \) can be computed as follows:

\[
T_w = L_w \quad \text{and} \quad \hat{e}_w = \frac{T_w}{l_w} = \frac{L_w}{d_w} = \frac{1}{M_w} \quad (3.12)
\]

In the 1970 California life table \( w = 85 \), and \( l_{85} = 23274 \). The death rate for age 85 and over is \( M_{85} = 0.157564 \); therefore
\[ L_{85} = \frac{\hat{\lambda}_{85}}{M_{85}} = \frac{23274}{.157564} = 147711 \]

and

\[ T_{85} = 147711 \quad \text{and} \quad \hat{e}_{85} = 6.35 \]
Figures 1 to 4 show graphically the probability of dying \( \hat{q}_x \), the number of survivors \( \hat{\ell}_x \), the number of deaths \( \hat{d}_x \), and the expectation of life \( \hat{e}_x \), for each age \( x \) for the total California population, 1970. Figures 5 to 8 show the corresponding four sets of quantities for the total United States population, 1970. As we see from Figure 1 that the probability of dying is extremely high for the first year of life. It decreases sharply after the first year and reaches a minimum at the age of 10 years. From there, the probability rises gradually and reaches the same magnitude of \( q_0 \) around age 65 and it continues to increase monotonically and later drastically. The pattern of \( \hat{q}_x \) is also reflected in the \( \hat{\ell}_x \), \( \hat{d}_x \) and \( \hat{e}_x \). Since the life tables for the California population and for the United States population end at age 85, the graphics also stop at that age.

In this manual we have introduced two sets of terms, one for the theoretical quantities and the other for the estimates of the theoretical quantities. The theoretical quantities include the probability of dying \( q_x \), the expectation of life \( e_x \) for each age \( x \), and many others; the corresponding estimates are \( \hat{q}_x \) and \( \hat{e}_x \). For simplicity of reading and when there would be no confusion, we shall drop the words "estimate of" and refer to \( \hat{q}_x \) as the probability of dying and \( \hat{e}_x \) as the expectation of life, etc.
Table 1.

Construction of Complete Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Midyear population in interval ((x,x+1))</th>
<th>Number of deaths in interval ((x,x+1))</th>
<th>Death rate in interval ((x,x+1))</th>
<th>Fraction of last year of life</th>
<th>Probability of dying in interval ((x,x+1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x ) to (x+1)</td>
<td>(P_x)</td>
<td>(D_x)</td>
<td>(M_x)</td>
<td>(a'_x)</td>
<td>(q_x)</td>
</tr>
<tr>
<td>0-1</td>
<td>340483</td>
<td>6234</td>
<td>.018309</td>
<td>.09</td>
<td>.01801</td>
</tr>
<tr>
<td>1-2</td>
<td>326154</td>
<td>368</td>
<td>.001128</td>
<td>.43</td>
<td>.00113</td>
</tr>
<tr>
<td>2-3</td>
<td>313699</td>
<td>269</td>
<td>.000858</td>
<td>.45</td>
<td>.00086</td>
</tr>
<tr>
<td>3-4</td>
<td>323441</td>
<td>237</td>
<td>.000733</td>
<td>.47</td>
<td>.00073</td>
</tr>
<tr>
<td>4-5</td>
<td>338904</td>
<td>175</td>
<td>.000516</td>
<td>.49</td>
<td>.00052</td>
</tr>
<tr>
<td>5-6</td>
<td>362161</td>
<td>179</td>
<td>.000494</td>
<td>.50</td>
<td>.00049</td>
</tr>
<tr>
<td>6-7</td>
<td>379642</td>
<td>171</td>
<td>.000450</td>
<td>.50</td>
<td>.00045</td>
</tr>
<tr>
<td>7-8</td>
<td>386980</td>
<td>131</td>
<td>.000339</td>
<td>.50</td>
<td>.00034</td>
</tr>
<tr>
<td>8-9</td>
<td>391610</td>
<td>121</td>
<td>.000309</td>
<td>.50</td>
<td>.00031</td>
</tr>
<tr>
<td>9-10</td>
<td>387724</td>
<td>121</td>
<td>.000304</td>
<td>.50</td>
<td>.00030</td>
</tr>
<tr>
<td>10-11</td>
<td>406118</td>
<td>126</td>
<td>.000310</td>
<td>.50</td>
<td>.00031</td>
</tr>
<tr>
<td>11-12</td>
<td>388927</td>
<td>127</td>
<td>.000327</td>
<td>.50</td>
<td>.00033</td>
</tr>
<tr>
<td>12-13</td>
<td>395025</td>
<td>138</td>
<td>.000349</td>
<td>.50</td>
<td>.00035</td>
</tr>
<tr>
<td>13-14</td>
<td>388526</td>
<td>158</td>
<td>.000407</td>
<td>.50</td>
<td>.00041</td>
</tr>
<tr>
<td>14-15</td>
<td>385085</td>
<td>186</td>
<td>.000483</td>
<td>.50</td>
<td>.00048</td>
</tr>
<tr>
<td>15-16</td>
<td>377127</td>
<td>235</td>
<td>.000623</td>
<td>.50</td>
<td>.00062</td>
</tr>
<tr>
<td>16-17</td>
<td>368156</td>
<td>344</td>
<td>.000934</td>
<td>.50</td>
<td>.00093</td>
</tr>
<tr>
<td>17-18</td>
<td>366198</td>
<td>385</td>
<td>.001051</td>
<td>.50</td>
<td>.00105</td>
</tr>
<tr>
<td>18-19</td>
<td>354932</td>
<td>506</td>
<td>.001426</td>
<td>.50</td>
<td>.00142</td>
</tr>
<tr>
<td>19-20</td>
<td>350966</td>
<td>584</td>
<td>.001664</td>
<td>.50</td>
<td>.00166</td>
</tr>
<tr>
<td>20-21</td>
<td>359833</td>
<td>583</td>
<td>.001620</td>
<td>.50</td>
<td>.00162</td>
</tr>
<tr>
<td>21-22</td>
<td>349557</td>
<td>562</td>
<td>.001608</td>
<td>.50</td>
<td>.00161</td>
</tr>
<tr>
<td>22-23</td>
<td>365839</td>
<td>572</td>
<td>.001564</td>
<td>.50</td>
<td>.00156</td>
</tr>
<tr>
<td>23-24</td>
<td>370548</td>
<td>564</td>
<td>.001522</td>
<td>.50</td>
<td>.00152</td>
</tr>
<tr>
<td>24-25</td>
<td>295189</td>
<td>421</td>
<td>.001426</td>
<td>.50</td>
<td>.00143</td>
</tr>
<tr>
<td>25-26</td>
<td>304013</td>
<td>416</td>
<td>.001368</td>
<td>.50</td>
<td>.00137</td>
</tr>
<tr>
<td>26-27</td>
<td>305558</td>
<td>391</td>
<td>.001280</td>
<td>.50</td>
<td>.00128</td>
</tr>
<tr>
<td>27-28</td>
<td>310554</td>
<td>461</td>
<td>.001484</td>
<td>.50</td>
<td>.00148</td>
</tr>
<tr>
<td>28-29</td>
<td>275897</td>
<td>411</td>
<td>.001490</td>
<td>.50</td>
<td>.00149</td>
</tr>
<tr>
<td>29-30</td>
<td>261592</td>
<td>392</td>
<td>.001499</td>
<td>.50</td>
<td>.00150</td>
</tr>
</tbody>
</table>
Table 1. (continued)

Construction of Complete Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>$x$ to $x+1$</th>
<th>$p_x$</th>
<th>$d_x$</th>
<th>$m_x$</th>
<th>$a_x$</th>
<th>$q_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30-31</td>
<td>264083</td>
<td>399</td>
<td>.001511</td>
<td>.50</td>
<td>.00151</td>
</tr>
<tr>
<td>31-32</td>
<td>247777</td>
<td>378</td>
<td>.001526</td>
<td>.50</td>
<td>.00152</td>
</tr>
<tr>
<td>32-33</td>
<td>241726</td>
<td>388</td>
<td>.001605</td>
<td>.50</td>
<td>.00160</td>
</tr>
<tr>
<td>33-34</td>
<td>232025</td>
<td>365</td>
<td>.001573</td>
<td>.50</td>
<td>.00157</td>
</tr>
<tr>
<td>34-35</td>
<td>233778</td>
<td>434</td>
<td>.001856</td>
<td>.50</td>
<td>.00185</td>
</tr>
<tr>
<td>35-36</td>
<td>234338</td>
<td>439</td>
<td>.001873</td>
<td>.50</td>
<td>.00187</td>
</tr>
<tr>
<td>36-37</td>
<td>224302</td>
<td>475</td>
<td>.002118</td>
<td>.50</td>
<td>.00212</td>
</tr>
<tr>
<td>37-38</td>
<td>228652</td>
<td>519</td>
<td>.002270</td>
<td>.50</td>
<td>.00227</td>
</tr>
<tr>
<td>38-39</td>
<td>226727</td>
<td>549</td>
<td>.002421</td>
<td>.50</td>
<td>.00242</td>
</tr>
<tr>
<td>39-40</td>
<td>235980</td>
<td>606</td>
<td>.002568</td>
<td>.50</td>
<td>.00256</td>
</tr>
<tr>
<td>40-41</td>
<td>249027</td>
<td>665</td>
<td>.002670</td>
<td>.50</td>
<td>.00267</td>
</tr>
<tr>
<td>41-42</td>
<td>232893</td>
<td>719</td>
<td>.003087</td>
<td>.50</td>
<td>.00308</td>
</tr>
<tr>
<td>42-43</td>
<td>239747</td>
<td>862</td>
<td>.003600</td>
<td>.50</td>
<td>.00359</td>
</tr>
<tr>
<td>43-44</td>
<td>238783</td>
<td>874</td>
<td>.003660</td>
<td>.50</td>
<td>.00365</td>
</tr>
<tr>
<td>44-45</td>
<td>248100</td>
<td>993</td>
<td>.004002</td>
<td>.50</td>
<td>.00399</td>
</tr>
<tr>
<td>45-46</td>
<td>253828</td>
<td>1140</td>
<td>.004491</td>
<td>.50</td>
<td>.00448</td>
</tr>
<tr>
<td>46-47</td>
<td>249857</td>
<td>1268</td>
<td>.005075</td>
<td>.50</td>
<td>.00506</td>
</tr>
<tr>
<td>47-48</td>
<td>247955</td>
<td>1362</td>
<td>.005493</td>
<td>.50</td>
<td>.00548</td>
</tr>
<tr>
<td>48-49</td>
<td>252137</td>
<td>1422</td>
<td>.005640</td>
<td>.50</td>
<td>.00562</td>
</tr>
<tr>
<td>49-50</td>
<td>242126</td>
<td>1530</td>
<td>.006319</td>
<td>.50</td>
<td>.00630</td>
</tr>
<tr>
<td>50-51</td>
<td>243799</td>
<td>1594</td>
<td>.006538</td>
<td>.50</td>
<td>.00652</td>
</tr>
<tr>
<td>51-52</td>
<td>220599</td>
<td>1710</td>
<td>.007752</td>
<td>.50</td>
<td>.00772</td>
</tr>
<tr>
<td>52-53</td>
<td>213448</td>
<td>1793</td>
<td>.008400</td>
<td>.50</td>
<td>.00837</td>
</tr>
<tr>
<td>53-54</td>
<td>203618</td>
<td>1870</td>
<td>.009184</td>
<td>.50</td>
<td>.00914</td>
</tr>
<tr>
<td>54-55</td>
<td>202388</td>
<td>1981</td>
<td>.008788</td>
<td>.50</td>
<td>.00974</td>
</tr>
<tr>
<td>55-56</td>
<td>201750</td>
<td>2217</td>
<td>.010989</td>
<td>.50</td>
<td>.01093</td>
</tr>
<tr>
<td>56-57</td>
<td>193828</td>
<td>2333</td>
<td>.012036</td>
<td>.50</td>
<td>.01196</td>
</tr>
<tr>
<td>57-58</td>
<td>187257</td>
<td>2483</td>
<td>.013260</td>
<td>.50</td>
<td>.01317</td>
</tr>
<tr>
<td>58-59</td>
<td>178602</td>
<td>2392</td>
<td>.013393</td>
<td>.50</td>
<td>.01330</td>
</tr>
<tr>
<td>59-60</td>
<td>171807</td>
<td>2517</td>
<td>.014650</td>
<td>.50</td>
<td>.01454</td>
</tr>
<tr>
<td>60-61</td>
<td>174613</td>
<td>2733</td>
<td>.015652</td>
<td>.50</td>
<td>.01553</td>
</tr>
<tr>
<td>61-62</td>
<td>157734</td>
<td>2743</td>
<td>.017390</td>
<td>.50</td>
<td>.01724</td>
</tr>
<tr>
<td>62-63</td>
<td>154174</td>
<td>2911</td>
<td>.018881</td>
<td>.50</td>
<td>.01870</td>
</tr>
<tr>
<td>63-64</td>
<td>144149</td>
<td>2968</td>
<td>.020590</td>
<td>.50</td>
<td>.02038</td>
</tr>
<tr>
<td>64-65</td>
<td>140100</td>
<td>2954</td>
<td>.021085</td>
<td>.50</td>
<td>.02086</td>
</tr>
<tr>
<td>65-66</td>
<td>135857</td>
<td>3391</td>
<td>.024960</td>
<td>.50</td>
<td>.02465</td>
</tr>
<tr>
<td>66-67</td>
<td>129386</td>
<td>3278</td>
<td>.025335</td>
<td>.50</td>
<td>.02502</td>
</tr>
<tr>
<td>67-68</td>
<td>123925</td>
<td>3352</td>
<td>.027049</td>
<td>.50</td>
<td>.02669</td>
</tr>
<tr>
<td>68-69</td>
<td>112574</td>
<td>3331</td>
<td>.029589</td>
<td>.50</td>
<td>.02916</td>
</tr>
<tr>
<td>69-70</td>
<td>119063</td>
<td>3736</td>
<td>.031378</td>
<td>.50</td>
<td>.03089</td>
</tr>
<tr>
<td>x to x+1</td>
<td>$p_x$</td>
<td>$d_x$</td>
<td>$n_x$</td>
<td>$a_x^{(')}$</td>
<td>$\delta_x$</td>
</tr>
<tr>
<td>----------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------------</td>
<td>----------</td>
</tr>
<tr>
<td>70-71</td>
<td>114066</td>
<td>3846</td>
<td>0.033717</td>
<td>0.50</td>
<td>0.03316</td>
</tr>
<tr>
<td>71-72</td>
<td>100781</td>
<td>3704</td>
<td>0.036753</td>
<td>0.50</td>
<td>0.03609</td>
</tr>
<tr>
<td>72-73</td>
<td>93031</td>
<td>3706</td>
<td>0.039836</td>
<td>0.50</td>
<td>0.03906</td>
</tr>
<tr>
<td>73-74</td>
<td>89992</td>
<td>3830</td>
<td>0.042559</td>
<td>0.50</td>
<td>0.04167</td>
</tr>
<tr>
<td>74-75</td>
<td>86561</td>
<td>4063</td>
<td>0.046938</td>
<td>0.50</td>
<td>0.04586</td>
</tr>
<tr>
<td>75-76</td>
<td>81003</td>
<td>4275</td>
<td>0.052776</td>
<td>0.50</td>
<td>0.05142</td>
</tr>
<tr>
<td>76-77</td>
<td>73552</td>
<td>4383</td>
<td>0.059590</td>
<td>0.50</td>
<td>0.05787</td>
</tr>
<tr>
<td>77-78</td>
<td>70516</td>
<td>4259</td>
<td>0.060398</td>
<td>0.50</td>
<td>0.05863</td>
</tr>
<tr>
<td>78-79</td>
<td>60616</td>
<td>4181</td>
<td>0.068975</td>
<td>0.50</td>
<td>0.06668</td>
</tr>
<tr>
<td>79-80</td>
<td>56410</td>
<td>4227</td>
<td>0.074934</td>
<td>0.50</td>
<td>0.07223</td>
</tr>
<tr>
<td>80-81</td>
<td>57646</td>
<td>4424</td>
<td>0.076744</td>
<td>0.50</td>
<td>0.07391</td>
</tr>
<tr>
<td>81-82</td>
<td>48299</td>
<td>4288</td>
<td>0.088780</td>
<td>0.50</td>
<td>0.08501</td>
</tr>
<tr>
<td>82-83</td>
<td>39560</td>
<td>3995</td>
<td>0.100986</td>
<td>0.50</td>
<td>0.09613</td>
</tr>
<tr>
<td>83-84</td>
<td>34439</td>
<td>3753</td>
<td>0.108975</td>
<td>0.50</td>
<td>0.10334</td>
</tr>
<tr>
<td>84-85</td>
<td>31009</td>
<td>3669</td>
<td>0.118320</td>
<td>0.50</td>
<td>0.11171</td>
</tr>
<tr>
<td>85+</td>
<td>142691</td>
<td>22483</td>
<td>0.157564</td>
<td>1.00000</td>
<td></td>
</tr>
</tbody>
</table>
### Table 2.
Complete Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Probability of dying in intervals (in years) (q_x)</th>
<th>Number living at age (x) (L_x)</th>
<th>Number dying in interval ((x,x+1)) (d_x)</th>
<th>Fraction of last year of life (a'_x)</th>
<th>Number of years lived in interval ((x,x+1)) (T_x)</th>
<th>Total number of years lived beyond age (x) (T_x)</th>
<th>Observed Expectation of life at age (x) (\hat{e}_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0.01801</td>
<td>100000</td>
<td>1801</td>
<td>0.09</td>
<td>98361</td>
<td>7190390</td>
<td>71.90</td>
</tr>
<tr>
<td>1-2</td>
<td>0.00113</td>
<td>98199</td>
<td>111</td>
<td>0.43</td>
<td>98136</td>
<td>7092029</td>
<td>72.22</td>
</tr>
<tr>
<td>2-3</td>
<td>0.00086</td>
<td>98088</td>
<td>84</td>
<td>0.45</td>
<td>98042</td>
<td>6993893</td>
<td>71.30</td>
</tr>
<tr>
<td>3-4</td>
<td>0.00073</td>
<td>98004</td>
<td>72</td>
<td>0.47</td>
<td>97966</td>
<td>6895851</td>
<td>70.36</td>
</tr>
<tr>
<td>4-5</td>
<td>0.00052</td>
<td>97932</td>
<td>51</td>
<td>0.49</td>
<td>97906</td>
<td>6797885</td>
<td>69.41</td>
</tr>
<tr>
<td>5-6</td>
<td>0.00049</td>
<td>97881</td>
<td>48</td>
<td>0.50</td>
<td>97857</td>
<td>6699979</td>
<td>68.45</td>
</tr>
<tr>
<td>6-7</td>
<td>0.00045</td>
<td>97833</td>
<td>44</td>
<td>0.50</td>
<td>97811</td>
<td>6602122</td>
<td>67.48</td>
</tr>
<tr>
<td>7-8</td>
<td>0.00034</td>
<td>97789</td>
<td>33</td>
<td>0.50</td>
<td>97772</td>
<td>6504311</td>
<td>66.51</td>
</tr>
<tr>
<td>8-9</td>
<td>0.00031</td>
<td>97756</td>
<td>30</td>
<td>0.50</td>
<td>97741</td>
<td>6406539</td>
<td>65.54</td>
</tr>
<tr>
<td>9-10</td>
<td>0.00030</td>
<td>97726</td>
<td>29</td>
<td>0.50</td>
<td>97711</td>
<td>6308798</td>
<td>64.56</td>
</tr>
<tr>
<td>10-11</td>
<td>0.00031</td>
<td>97697</td>
<td>30</td>
<td>0.50</td>
<td>97682</td>
<td>6211087</td>
<td>63.58</td>
</tr>
<tr>
<td>11-12</td>
<td>0.00033</td>
<td>97667</td>
<td>32</td>
<td>0.50</td>
<td>97651</td>
<td>6113405</td>
<td>62.59</td>
</tr>
<tr>
<td>12-13</td>
<td>0.00035</td>
<td>97635</td>
<td>34</td>
<td>0.50</td>
<td>97618</td>
<td>6015754</td>
<td>61.61</td>
</tr>
<tr>
<td>13-14</td>
<td>0.00041</td>
<td>97601</td>
<td>40</td>
<td>0.50</td>
<td>97581</td>
<td>5918136</td>
<td>60.64</td>
</tr>
<tr>
<td>14-15</td>
<td>0.00048</td>
<td>97561</td>
<td>47</td>
<td>0.50</td>
<td>97538</td>
<td>5820555</td>
<td>59.66</td>
</tr>
<tr>
<td>15-16</td>
<td>0.00062</td>
<td>97514</td>
<td>60</td>
<td>0.50</td>
<td>97484</td>
<td>5723017</td>
<td>58.69</td>
</tr>
<tr>
<td>16-17</td>
<td>0.00093</td>
<td>97454</td>
<td>91</td>
<td>0.50</td>
<td>97408</td>
<td>5625533</td>
<td>57.73</td>
</tr>
<tr>
<td>17-18</td>
<td>0.00105</td>
<td>97363</td>
<td>102</td>
<td>0.50</td>
<td>97312</td>
<td>5528125</td>
<td>56.78</td>
</tr>
<tr>
<td>18-19</td>
<td>0.00142</td>
<td>97261</td>
<td>138</td>
<td>0.50</td>
<td>97192</td>
<td>5430813</td>
<td>55.84</td>
</tr>
<tr>
<td>19-20</td>
<td>0.00166</td>
<td>97123</td>
<td>161</td>
<td>0.50</td>
<td>97043</td>
<td>5333621</td>
<td>54.92</td>
</tr>
<tr>
<td>20-21</td>
<td>0.00162</td>
<td>96962</td>
<td>157</td>
<td>0.50</td>
<td>96884</td>
<td>5236578</td>
<td>54.01</td>
</tr>
<tr>
<td>21-22</td>
<td>0.00161</td>
<td>96805</td>
<td>156</td>
<td>0.50</td>
<td>96727</td>
<td>5139694</td>
<td>53.09</td>
</tr>
<tr>
<td>22-23</td>
<td>0.00156</td>
<td>96649</td>
<td>151</td>
<td>0.50</td>
<td>96574</td>
<td>5042967</td>
<td>52.18</td>
</tr>
<tr>
<td>23-24</td>
<td>0.00152</td>
<td>96498</td>
<td>147</td>
<td>0.50</td>
<td>96424</td>
<td>4946393</td>
<td>51.26</td>
</tr>
<tr>
<td>24-25</td>
<td>0.00143</td>
<td>96351</td>
<td>138</td>
<td>0.50</td>
<td>96282</td>
<td>4849969</td>
<td>50.34</td>
</tr>
<tr>
<td>25-26</td>
<td>0.00137</td>
<td>96213</td>
<td>132</td>
<td>0.50</td>
<td>96147</td>
<td>4753687</td>
<td>49.41</td>
</tr>
<tr>
<td>26-27</td>
<td>0.00128</td>
<td>96081</td>
<td>123</td>
<td>0.50</td>
<td>96020</td>
<td>4657540</td>
<td>48.48</td>
</tr>
<tr>
<td>27-28</td>
<td>0.00148</td>
<td>95958</td>
<td>142</td>
<td>0.50</td>
<td>95887</td>
<td>4561520</td>
<td>47.54</td>
</tr>
<tr>
<td>28-29</td>
<td>0.00149</td>
<td>95816</td>
<td>143</td>
<td>0.50</td>
<td>95745</td>
<td>4465633</td>
<td>46.61</td>
</tr>
<tr>
<td>29-30</td>
<td>0.00150</td>
<td>95673</td>
<td>144</td>
<td>0.50</td>
<td>95601</td>
<td>4369888</td>
<td>45.68</td>
</tr>
</tbody>
</table>
Table 2. (continued)
Complete Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Probability of dying in interval (x,x+1)</th>
<th>Number living at age x</th>
<th>Number dying in interval (x,x+1)</th>
<th>Fraction of last year of life</th>
<th>Number of years lived in interval (x,x+1)</th>
<th>Total number of years lived beyond age x</th>
<th>Observed Expectation of life at age x</th>
</tr>
</thead>
<tbody>
<tr>
<td>x to x+1</td>
<td>q_x</td>
<td>x</td>
<td>d_x</td>
<td>a'x</td>
<td>L_x</td>
<td>T_x</td>
<td>e_x</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
</tr>
<tr>
<td>30-31</td>
<td>.00151</td>
<td>95529</td>
<td>144</td>
<td>.50</td>
<td>95457</td>
<td>4274287</td>
<td>44.74</td>
</tr>
<tr>
<td>31-32</td>
<td>.00152</td>
<td>95385</td>
<td>145</td>
<td>.50</td>
<td>95312</td>
<td>4178830</td>
<td>43.81</td>
</tr>
<tr>
<td>32-33</td>
<td>.00160</td>
<td>95240</td>
<td>152</td>
<td>.50</td>
<td>95164</td>
<td>408318</td>
<td>42.88</td>
</tr>
<tr>
<td>33-34</td>
<td>.00157</td>
<td>95088</td>
<td>149</td>
<td>.50</td>
<td>95014</td>
<td>3988354</td>
<td>41.94</td>
</tr>
<tr>
<td>34-35</td>
<td>.00185</td>
<td>94939</td>
<td>176</td>
<td>.50</td>
<td>94851</td>
<td>3893340</td>
<td>41.01</td>
</tr>
<tr>
<td>35-36</td>
<td>.00187</td>
<td>94763</td>
<td>177</td>
<td>.50</td>
<td>94674</td>
<td>3798489</td>
<td>40.08</td>
</tr>
<tr>
<td>36-37</td>
<td>.00212</td>
<td>94586</td>
<td>201</td>
<td>.50</td>
<td>94486</td>
<td>3703815</td>
<td>39.16</td>
</tr>
<tr>
<td>37-38</td>
<td>.00227</td>
<td>94385</td>
<td>214</td>
<td>.50</td>
<td>94278</td>
<td>3609329</td>
<td>38.24</td>
</tr>
<tr>
<td>38-39</td>
<td>.00242</td>
<td>94171</td>
<td>228</td>
<td>.50</td>
<td>94057</td>
<td>3515051</td>
<td>37.33</td>
</tr>
<tr>
<td>39-40</td>
<td>.00256</td>
<td>93943</td>
<td>240</td>
<td>.50</td>
<td>93823</td>
<td>3420994</td>
<td>36.42</td>
</tr>
<tr>
<td>40-41</td>
<td>.00267</td>
<td>93703</td>
<td>250</td>
<td>.50</td>
<td>93578</td>
<td>3327171</td>
<td>35.51</td>
</tr>
<tr>
<td>41-42</td>
<td>.00308</td>
<td>93453</td>
<td>288</td>
<td>.50</td>
<td>93309</td>
<td>3233593</td>
<td>34.60</td>
</tr>
<tr>
<td>42-43</td>
<td>.00359</td>
<td>93165</td>
<td>334</td>
<td>.50</td>
<td>92998</td>
<td>3140284</td>
<td>33.71</td>
</tr>
<tr>
<td>43-44</td>
<td>.00365</td>
<td>92831</td>
<td>339</td>
<td>.50</td>
<td>92661</td>
<td>3047286</td>
<td>32.83</td>
</tr>
<tr>
<td>44-45</td>
<td>.00399</td>
<td>92492</td>
<td>369</td>
<td>.50</td>
<td>92307</td>
<td>2954625</td>
<td>31.94</td>
</tr>
<tr>
<td>45-46</td>
<td>.00448</td>
<td>92123</td>
<td>413</td>
<td>.50</td>
<td>91916</td>
<td>2862318</td>
<td>31.07</td>
</tr>
<tr>
<td>46-47</td>
<td>.00506</td>
<td>91710</td>
<td>464</td>
<td>.50</td>
<td>91478</td>
<td>2770402</td>
<td>30.21</td>
</tr>
<tr>
<td>47-48</td>
<td>.00548</td>
<td>91246</td>
<td>500</td>
<td>.50</td>
<td>90996</td>
<td>2678924</td>
<td>29.36</td>
</tr>
<tr>
<td>48-49</td>
<td>.00562</td>
<td>90746</td>
<td>510</td>
<td>.50</td>
<td>90491</td>
<td>2587928</td>
<td>28.52</td>
</tr>
<tr>
<td>49-50</td>
<td>.00630</td>
<td>90236</td>
<td>568</td>
<td>.50</td>
<td>89952</td>
<td>2497437</td>
<td>27.68</td>
</tr>
<tr>
<td>50-51</td>
<td>.00652</td>
<td>89668</td>
<td>585</td>
<td>.50</td>
<td>89376</td>
<td>2407485</td>
<td>26.85</td>
</tr>
<tr>
<td>51-52</td>
<td>.00772</td>
<td>89083</td>
<td>688</td>
<td>.50</td>
<td>88739</td>
<td>2318109</td>
<td>26.02</td>
</tr>
<tr>
<td>52-53</td>
<td>.00837</td>
<td>88395</td>
<td>740</td>
<td>.50</td>
<td>88025</td>
<td>2229370</td>
<td>25.22</td>
</tr>
<tr>
<td>53-54</td>
<td>.00914</td>
<td>87655</td>
<td>801</td>
<td>.50</td>
<td>87255</td>
<td>2141345</td>
<td>24.43</td>
</tr>
<tr>
<td>54-55</td>
<td>.00974</td>
<td>86854</td>
<td>846</td>
<td>.50</td>
<td>86431</td>
<td>2054090</td>
<td>23.65</td>
</tr>
<tr>
<td>55-56</td>
<td>.01093</td>
<td>86008</td>
<td>940</td>
<td>.50</td>
<td>85538</td>
<td>1967659</td>
<td>22.88</td>
</tr>
<tr>
<td>56-57</td>
<td>.01196</td>
<td>85068</td>
<td>1017</td>
<td>.50</td>
<td>84559</td>
<td>1882121</td>
<td>22.12</td>
</tr>
<tr>
<td>57-58</td>
<td>.01317</td>
<td>84051</td>
<td>1107</td>
<td>.50</td>
<td>83497</td>
<td>1797562</td>
<td>21.39</td>
</tr>
<tr>
<td>58-59</td>
<td>.01330</td>
<td>82944</td>
<td>1103</td>
<td>.50</td>
<td>82393</td>
<td>1714065</td>
<td>20.67</td>
</tr>
<tr>
<td>59-60</td>
<td>.01454</td>
<td>81841</td>
<td>1190</td>
<td>.50</td>
<td>81246</td>
<td>1631672</td>
<td>19.94</td>
</tr>
</tbody>
</table>
Table 2. (continued)

Complete Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Probability of dying in interval (x,x+l) ( q_x )</th>
<th>Number dying in interval living at age ( x ) ( d_x )</th>
<th>Fraction of last year of life ( a'_x )</th>
<th>Number of years lived ( L_x )</th>
<th>Number of years lived beyond age ( x ) ( T_x )</th>
<th>Observed Expectation of life at age ( x ) ( \hat{e}_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60-61</td>
<td>.01553</td>
<td>80651</td>
<td>1253</td>
<td>.50</td>
<td>80025</td>
<td>155042</td>
</tr>
<tr>
<td>61-62</td>
<td>.01724</td>
<td>79398</td>
<td>1369</td>
<td>.50</td>
<td>78713</td>
<td>147040</td>
</tr>
<tr>
<td>62-63</td>
<td>.01870</td>
<td>78029</td>
<td>1459</td>
<td>.50</td>
<td>77299</td>
<td>139168</td>
</tr>
<tr>
<td>63-64</td>
<td>.02038</td>
<td>76570</td>
<td>1560</td>
<td>.50</td>
<td>75790</td>
<td>131438</td>
</tr>
<tr>
<td>64-65</td>
<td>.02086</td>
<td>75010</td>
<td>1565</td>
<td>.50</td>
<td>74228</td>
<td>123859</td>
</tr>
<tr>
<td>65-66</td>
<td>.02465</td>
<td>73445</td>
<td>1810</td>
<td>.50</td>
<td>72540</td>
<td>116437</td>
</tr>
<tr>
<td>66-67</td>
<td>.02502</td>
<td>71635</td>
<td>1792</td>
<td>.50</td>
<td>70739</td>
<td>109183</td>
</tr>
<tr>
<td>67-68</td>
<td>.02669</td>
<td>69843</td>
<td>1864</td>
<td>.50</td>
<td>68911</td>
<td>102109</td>
</tr>
<tr>
<td>68-69</td>
<td>.02916</td>
<td>67979</td>
<td>1982</td>
<td>.50</td>
<td>66988</td>
<td>95218</td>
</tr>
<tr>
<td>69-70</td>
<td>.03089</td>
<td>65997</td>
<td>2039</td>
<td>.50</td>
<td>64978</td>
<td>88519</td>
</tr>
<tr>
<td>70-71</td>
<td>.03316</td>
<td>63958</td>
<td>2121</td>
<td>.50</td>
<td>62897</td>
<td>820215</td>
</tr>
<tr>
<td>71-72</td>
<td>.03609</td>
<td>61837</td>
<td>2232</td>
<td>.50</td>
<td>60721</td>
<td>757318</td>
</tr>
<tr>
<td>72-73</td>
<td>.03906</td>
<td>59605</td>
<td>2328</td>
<td>.50</td>
<td>58441</td>
<td>696597</td>
</tr>
<tr>
<td>73-74</td>
<td>.04167</td>
<td>57277</td>
<td>2387</td>
<td>.50</td>
<td>56083</td>
<td>638156</td>
</tr>
<tr>
<td>74-75</td>
<td>.04586</td>
<td>54890</td>
<td>2517</td>
<td>.50</td>
<td>53632</td>
<td>582073</td>
</tr>
<tr>
<td>75-76</td>
<td>.05142</td>
<td>52373</td>
<td>2693</td>
<td>.50</td>
<td>51026</td>
<td>528441</td>
</tr>
<tr>
<td>76-77</td>
<td>.05787</td>
<td>49680</td>
<td>2875</td>
<td>.50</td>
<td>48243</td>
<td>477415</td>
</tr>
<tr>
<td>77-78</td>
<td>.05863</td>
<td>46805</td>
<td>2744</td>
<td>.50</td>
<td>45433</td>
<td>429172</td>
</tr>
<tr>
<td>78-79</td>
<td>.06668</td>
<td>44061</td>
<td>2938</td>
<td>.50</td>
<td>42592</td>
<td>383739</td>
</tr>
<tr>
<td>79-80</td>
<td>.07223</td>
<td>41123</td>
<td>2970</td>
<td>.50</td>
<td>39638</td>
<td>341147</td>
</tr>
<tr>
<td>80-81</td>
<td>.07391</td>
<td>38153</td>
<td>2820</td>
<td>.50</td>
<td>36743</td>
<td>301509</td>
</tr>
<tr>
<td>81-82</td>
<td>.08501</td>
<td>35333</td>
<td>3004</td>
<td>.50</td>
<td>33831</td>
<td>264766</td>
</tr>
<tr>
<td>82-83</td>
<td>.09613</td>
<td>32329</td>
<td>3108</td>
<td>.50</td>
<td>30775</td>
<td>230935</td>
</tr>
<tr>
<td>83-84</td>
<td>.10334</td>
<td>29221</td>
<td>3020</td>
<td>.50</td>
<td>27711</td>
<td>200160</td>
</tr>
<tr>
<td>84-85</td>
<td>.11171</td>
<td>26201</td>
<td>2927</td>
<td>.50</td>
<td>24738</td>
<td>172449</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>23274</td>
<td>23274</td>
<td>.50</td>
<td>147711</td>
<td>147711</td>
</tr>
</tbody>
</table>
FIGURE 1. PROBABILITY OF DYING
TOTAL CALIFORNIA POPULATION, 1970
FIGURE 2. NUMBER OF SURVIVORS OUT OF 100,000 LIVE BIRTHS
TOTAL CALIFORNIA POPULATION, 1970
FIGURE 3. NUMBER OF DEATHS OUT OF 100,000 LIVE BIRTHS
TOTAL CALIFORNIA POPULATION, 1970

AGE AT DEATH IN YEARS, x

NUMBER OF DEATHS, d_x

0 10 20 30 40 50 60 70 80 90
0 500 1,000 1,500 2,000 2,500 3,000
FIGURE 4. EXPECTATION OF LIFE
TOTAL CALIFORNIA POPULATION, 1970

EXPECTATION OF LIFE AT AGE $x$, $\hat{e}_x$

AGE IN YEARS, $x$
FIGURE 5. PROBABILITY OF DYING
TOTAL UNITED STATES POPULATION, 1970

PROBABILITY OF DYING, 10000 \( q_x \)

AGE AT DEATH IN YEARS, \( x \)
FIGURE 6. NUMBER OF SURVIVORS OUT OF 100,000 LIVE BIRTHS
TOTAL UNITED STATES POPULATION, 1970

AGE IN YEARS, x

NUMBER OF SURVIVORS, A_x
FIGURE 7. NUMBER OF DEATHS OUT OF 100,000 LIVE BIRTHS
TOTAL UNITED STATES POPULATION, 1970

NUMBER OF DEATHS, $d_x$

AGE IN YEARS, $x$
Figure 8. Expectation of Life
Total United States Population, 1970
CHAPTER 5
THE LIFE TABLE AND ITS CONSTRUCTION – ABRIDGED LIFE TABLES

1. Introduction

Clearly, the current life table furnishes information not obtainable from other sources. It provides the public health worker, demographers and other research workers with tools for making international comparisons as well as for comparing contemporary groups within a country or for assessing trends within a given population. The life table death rate has the advantage over other mortality indices of being independent of age and sex distributions. This, of course, is also true for $\hat{e}_0$, the average length of life, or for $\hat{e}_x$, the average remaining lifetime at any age $x$. The ratio $\frac{\lambda_k}{\lambda_j}$ gives a convenient measurement for comparing the survival of selected age segments of two populations; for example, one might want to know if Swedish women who survive to age 20 have as good a chance of surviving to age 45 as do their Italian counterparts by comparing the ratios $\frac{\lambda_{45}}{\lambda_{20}}$.

Life table estimates have the disadvantage of any statistics based on the population census and vital records. Individuals or entire households may be missed by the census taker or overlooked by the informant. Cross-checking with birth and death certificates show that young children, even when they survive infancy, are sometimes forgotten; migratory segments of the population (particularly young males) are subject to marked under enumeration. Misstatements of age are clearly discernible in bar graphs of the age distribution particularly the overstatement of the ages of young children (followed by an understatement in the middle years), of persons approaching retirement age, and of the very old; in addition, a heaping is found for ages in multiples of five and at even ages.
Completeness of birth registration varies from country to country and must occasionally be checked. Death registration can be improved by the requirement that it be filed before a burial permit is issued. These defects in mortality data and population census have a marked effect on the complete life table.

There are three other disadvantages of complete life tables that are more closely related to the tables themselves. (1) The data necessary for intervals of one year of age is frequently not available; (2) Computations are tedious and time-consuming when computer services are not available; (3) A table consisting of 85 or 95 age groups does not present a concise picture of the mortality experience of a population.

These objections can be obviated by constructing an abridged rather than the complete life table. The computations are discussed in the following paragraphs.

2. A Method of Life Table Construction

An abridged life table contains columns similar to those described for the complete life table. The limits of age intervals are denoted by \( x_i \), \( i=0,1,\ldots,w \), and the length of the interval by \( n_i \) so that \( x_{i+1} - x_i = n_i \). Thus we have

- **Column 1.** Age interval \((x_i, x_{i+1})\)
- **Column 2.** Proportion dying in interval \((x_i, x_{i+1})\), \( q_i \).
- **Column 3.** Number alive at age \( x_i \), \( l_i \).
- **Column 4.** Number dying in interval \((x_i, x_{i+1})\), \( d_i \).
- **Column 5.** The average fraction of interval \((x_i, x_{i+1})\) lived by an individual dying at an age included in the interval \( a_i \).
- **Column 6.** Total number of years lived in interval \((x_i, x_{i+1})\), \( L_i \).
- **Column 7.** Total number of years lived beyond age \( x_i \), \( T_i \).
- **Column 8.** Observed expectation of life age at \( x_i \), \( e_i \).

The present method of constructing the abridged life table was proposed by Chiang [1960b], [1961] and was used for the 1959-61 California
abridged Life Tables [Norris]. The idea and procedure involved are the same as those used in the construction of the complete life table described in Chapter 4 with differences due only to the length of intervals. The length of the typical interval \( (x_i, x_{i+1}) \) in the abridged table is \( n_i = x_{i+1} - x_i \), which is greater than one year (commonly, \( n_i = 5 \) years, see Table 2). The essential element here is the average fraction of the interval lived by each person who dies at an age included in the interval. This fraction, called the fraction of last age interval of life, denoted by \( a_i \), is conceptually a logical extension of the fraction of the last year of life, \( a' \). Determination and discussion of \( a_i \) will be presented in Section 4. We use \( a_i \) as the point of departure.

Starting with the values of \( a_i \) we can construct the abridged life table by following the steps in Chapter 4. Because of its importance, however, we repeat the previous argument to derive the formula for \( q_i \), the estimate of the probability that an individual alive at age \( x_i \) will die in the interval \( (x_i, x_{i+1}) \). Let \( D_i \) be the number of deaths occurring in the age interval \( (x_i, x_{i+1}) \) during the calendar year under consideration, \( m_i \) the corresponding age-specific death rate. To derive a relationship between \( q_i \) and \( m_i \), we introduce \( b_i \), the number of individuals alive at exact age \( x_i \), such that among the \( b_i \) persons \( b_i \) will die in the interval. Then by definition the proportion dying in \( (x_i, x_{i+1}) \) is given by

\[
\hat{q}_i = \frac{b_i}{n_i}.
\]

(2.1)

The age specific death rate \( m_i \) is the ratio of \( b_i \) to the total number of years lived by the \( n_i \) individuals during the interval \( (x_i, x_{i+1}) \), or

\[
\hat{m}_i = \frac{b_i}{(x_{i+1} - x_i)n_i} + \frac{n_i}{n_i}.
\]

(2.2)
The first term in the denominator of (2.2) is the number of years lived by the \( (N_i - D_i) \) survivors, while the second term is the number of years lived by those who die in \((x_i, x_{i+1})\). Eliminating \( N_i \) from (2.1) and (2.2) yields the basic formula in the construction of an abridged life table

\[
\hat{n}_i = \frac{n_i M_i}{1 + (1 - n_i) n_i M_i} \quad (2.3)
\]

The age-specific death rate \( M_i \) may be estimated from

\[
M_i = \frac{d_i}{P_i} \quad (2.4)
\]

with \( P_i \) being the mid-year population.

All other quantities in the table are functions of \( \hat{n}_i \), \( a_i \) and the radix \( l_0 \). The number \( d_i \) of deaths in \((x_i, x_{i+1})\) and the number \( \ell_{i+1} \) of survivors at age \( x_{i+1} \) are computed from

\[
d_i = \ell_i \hat{n}_i, \quad i=0,1,\ldots,w-1, \quad (2.5)
\]

and

\[
\ell_{i+1} = \ell_i - d_i, \quad i=0,1,\ldots,w-1, \quad (2.6)
\]

respectively. The number of years lived in the interval \((x_i, x_{i+1})\) by the \( \ell_i \) survivors at age \( x_i \) is

\[
L_i = n_i (\ell_i - d_i) + a_i n_i d_i, \quad i=0,1,\ldots,w-1. \quad (2.7)
\]

The final age interval is again an open interval, and \( L_w \) is computed exactly as in the complete life table [cf., equation (3.6) in Chapter 4]:
\[ l_x = \frac{l_x}{l_x}, \quad (2.8) \]

where \( l_x \) is again the specific death rate for people of age \( x \) and over.

The total number \( T_i \) of years remaining to all the people attaining age \( x_i \) is the sum of \( L_j \) for \( j = i, i+1, \ldots, w \). The observed expectation of life \( \hat{e}_i \) at age \( x_i \) is the ratio \( T_i/l_i \), or

\[ \hat{e}_i = \frac{L_i + L_{i+1} + \ldots + L_w}{l_i}, \quad i = 0, \ldots, w. \quad (2.9) \]

As an example, the abridged life table for the California 1970 total population is given in Tables 1 and 2. The required data for constructing an abridged life table is the death rate \( (M_i) \) and the fraction of last age interval of life \( (a_i) \) for each age group. The death rate may be computed from the mid-year population \( (P_i \) column (2) in Table 1) and the number of deaths \( (D_i \), column (3)) of the population in question using formula (2.4). For the California 1970 total population, for example, the death rate \( M_0 \) is computed from

\[ M_0 = \frac{D_0}{P_0} = \frac{6234}{340483} = 0.018309. \quad (2.4a) \]

The fraction of the last age interval of life, \( a_i \), remains relatively constant over time for a given age interval \( (x_i, x_{i+1}) \). The only exception is \( a_0 \), which may be computed from the readily available published data on infant deaths. The value of \( a_i \), for \( i = 0, 1, \ldots \), have been computed for several countries and are given in Appendix V. They may be revised every 10 years.

When death rate \( (M_i) \) for each age group of a population is determined, one uses the corresponding \( a_i \) and formula (2.3) to compute \( \hat{e}_i \). The figures in other columns in the life table can be obtained by using formulas (2.5) through (2.9). The reader should use these formulas to verify the numerical values in Tables 1 and 2.
Table 1.
Construction of Abridged Life Table for Total California Population, USA, 1970

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Mid-year population in interval $(x_i, x_{i+1})$</th>
<th>Number of deaths in interval $(x_i, x_{i+1})$</th>
<th>Fraction of last age interval of life</th>
<th>Probability of dying in interval $(x_i, x_{i+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$ to $x_{i+1}$</td>
<td>$p_i$</td>
<td>$d_i$</td>
<td>$n_i$</td>
<td>$a_i$</td>
</tr>
<tr>
<td>0-1</td>
<td>340483</td>
<td>6234</td>
<td>.018309</td>
<td>.09</td>
</tr>
<tr>
<td>1-5</td>
<td>1302198</td>
<td>1049</td>
<td>.000806</td>
<td>.41</td>
</tr>
<tr>
<td>5-10</td>
<td>1918117</td>
<td>723</td>
<td>.000377</td>
<td>.44</td>
</tr>
<tr>
<td>10-15</td>
<td>1963681</td>
<td>735</td>
<td>.000374</td>
<td>.54</td>
</tr>
<tr>
<td>15-20</td>
<td>1817379</td>
<td>2054</td>
<td>.001130</td>
<td>.59</td>
</tr>
<tr>
<td>20-25</td>
<td>1740966</td>
<td>2702</td>
<td>.001552</td>
<td>.49</td>
</tr>
<tr>
<td>25-30</td>
<td>1457614</td>
<td>2071</td>
<td>.001421</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>1219389</td>
<td>1964</td>
<td>.001611</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>1149999</td>
<td>2588</td>
<td>.002250</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>1208550</td>
<td>4114</td>
<td>.003404</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>1245903</td>
<td>6722</td>
<td>.005395</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>1083852</td>
<td>8948</td>
<td>.008256</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>933244</td>
<td>11942</td>
<td>.012796</td>
<td>.52</td>
</tr>
<tr>
<td>60-65</td>
<td>770770</td>
<td>14309</td>
<td>.018565</td>
<td>.52</td>
</tr>
<tr>
<td>65-70</td>
<td>620805</td>
<td>17088</td>
<td>.027526</td>
<td>.51</td>
</tr>
<tr>
<td>70-75</td>
<td>484431</td>
<td>19149</td>
<td>.039529</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>342097</td>
<td>21325</td>
<td>.062336</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>210953</td>
<td>20129</td>
<td>.095419</td>
<td>.50</td>
</tr>
<tr>
<td>85+</td>
<td>142691</td>
<td>22483</td>
<td>.157564</td>
<td></td>
</tr>
<tr>
<td>Age interval (in years)</td>
<td>Probability of dying in interval $(x_i, x_{i+1})$</td>
<td>Number living at age $x_i$</td>
<td>Number dying in interval $(x_i, x_{i+1})$</td>
<td>Fraction of last year of life</td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------------------------------------</td>
<td>---------------------------</td>
<td>------------------------------------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>0-1</td>
<td>.01801</td>
<td>100000</td>
<td>1801</td>
<td>.09</td>
</tr>
<tr>
<td>1-5</td>
<td>.00322</td>
<td>98199</td>
<td>316</td>
<td>.41</td>
</tr>
<tr>
<td>5-10</td>
<td>.00188</td>
<td>97883</td>
<td>184</td>
<td>.44</td>
</tr>
<tr>
<td>10-15</td>
<td>.00187</td>
<td>97699</td>
<td>183</td>
<td>.54</td>
</tr>
<tr>
<td>15-20</td>
<td>.00564</td>
<td>97516</td>
<td>550</td>
<td>.59</td>
</tr>
<tr>
<td>20-25</td>
<td>.00773</td>
<td>96966</td>
<td>750</td>
<td>.49</td>
</tr>
<tr>
<td>25-30</td>
<td>.00708</td>
<td>96216</td>
<td>681</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>.00802</td>
<td>95535</td>
<td>766</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.01119</td>
<td>94769</td>
<td>160</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.01689</td>
<td>93709</td>
<td>1583</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.02664</td>
<td>92126</td>
<td>2454</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.04049</td>
<td>89672</td>
<td>3631</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.06207</td>
<td>86041</td>
<td>5341</td>
<td>.52</td>
</tr>
<tr>
<td>60-65</td>
<td>.08886</td>
<td>80700</td>
<td>7171</td>
<td>.52</td>
</tr>
<tr>
<td>65-70</td>
<td>.12893</td>
<td>73529</td>
<td>9480</td>
<td>.51</td>
</tr>
<tr>
<td>70-75</td>
<td>.18052</td>
<td>64049</td>
<td>11562</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>.27039</td>
<td>52487</td>
<td>14192</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.38521</td>
<td>38295</td>
<td>14752</td>
<td>.50</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>23543</td>
<td>23543</td>
<td></td>
</tr>
</tbody>
</table>
3. The Fraction of the Last Year of Life, \(a'_x\), and the Fraction of the Last Age Interval of Life, \(a_i\)

3.1 The fraction of the last year of life, \(a'_x\). The description of life table construction in the preceding section clearly indicates that the main ingredient in the construction of complete life tables is the fraction of the last year of life. Computation of the fraction is quite easy when the necessary data are available. Since we need to know only the exact number of days lived past the final birthday, which may be obtained from the date of birth and the date of death. In the State of California, both dates are key punched into tabulation cards, from which a computer can make the required subtraction to get the number of days lived during the last year of life for each person who died, sum the number of days lived, divide by the number of deaths to obtain the average number of days lived, and divide by 365 (or 366) days to give the desired fraction of year lived, \(a'_x\). Various statistical tests have been performed regarding the fraction \(a'_x\) using 132,205 California resident death records in 1960. Some of the results are briefly described below.

First the hypothesis of uniform distribution of deaths was tested for each year of age. The number of deaths by days lived during the last year of life was tabulated for each race and sex group and for each year of age. The year was divided into 26 intervals of 14 days each, except for the first interval which was 15 days. A frequency distribution of the number of deaths by intervals of days lived for each of the selected ages is shown in Table 5. The Chi-square method was used to test the uniform distribution of deaths. For the 10 ages shown, Chi-square values were significant for ages 0, 1, and 59.
The distribution of the deaths in the first year of life is highly skewed with the first interval of 15 days accounting for almost 70 percent of the total 8624 deaths. The second interval of 14 days contains only about 3.5 percent of the deaths, and this percentage decreases with the increase in age. The distribution of deaths in the second year also shows a decrease in the percentage of deaths with increasing age, although in a much smaller degree. No definite pattern can be ascertained for the distribution of deaths for age 59.

The t-test and the F-test were also performed for the difference between the observed fraction \(a'_x\) and the hypothesized value of .5, and for the difference between sex and race groups for each year of life. The results show that from age 5 on, the fraction \(a'_x\) is invariant with respect to sex and race and the assumed value of .5 is accepted. For the first 5 years of life, the data suggest the values of \(a'_0 = .09, a'_1 = .43, a'_2 = .45, a'_3 = .47,\) and \(a'_4 = .49\) for both sexes. These values, except for \(a'_0\), may be assumed for other countries. The value of \(a'_0\), however, needs to be computed for each country. The data required for the computation of \(a'_0\) are usually available in vital statistics publications.

Computation of \(a'_0\) is shown in Table 6, where the 1970 United States infant death data are used for illustration. The number of deaths in column (3) by age at death are usually available in vital statistics publications. The average point for each interval [Column (2)], takes into account the distribution of deaths in each interval. The product of the figures in columns (2) and (3), recorded in column (4), is the number of days lived by individuals who died in each interval. The sum of the products, appearing in the lower right hand corner (2,464,403,7 in this case) is the total number of days lived by the (74,667) infants who
died during the first year of life. This total, when divided by 74,667 \times 365, gives the fraction of \( a_0' \), the fraction of the year lived by an infant who dies during the first year of life. Since both the complete life table and the abridged life table begin with the age interval (0,1), \( a_0 = a_0' \).

3.2 The fraction of the last age interval of life, \( a_1' \). This fraction is as essential in the construction of the abridged life table as the fraction of the last year of life in the complete table. Conceptually, it is an extension of the latter. When a person dies at age 23, for example, he has lived a certain fraction of the age interval (20, 25). The average fraction lived in each interval \((x_i, x_{i+1})\), which is called the fraction of the last age interval of life, depends on the probability of dying and the corresponding fraction of last year of life \( a_x' \) for each year of age within the interval. The relation between \( a_1 \) and the probabilities \( q_x \) and \( p_x (= 1-q_x) \) and \( a_x' \) is derived as follows.

Consider the age interval \((1, 5)\) and the fraction of the interval \( a_1 \) that a person will live if he dies between ages one and five. For a person alive at the exact age of one year [i.e., the beginning of the interval \((1, 5)\)], there is a probability \( q_1 \) that he will die during the year \((1, 2)\), a probability \((1-q_1)q_2 = p_1q_2\) that he will die in \((2, 3)\), a probability \(p_1p_2q_3\) of dying in \((3, 4)\), and a probability \(p_1p_2p_3q_4\) of dying in \((4, 5)\). The corresponding periods of time that he lives are \(a_1'\), \((1+a_2')\), \((2+a_3')\), and \((3+a_4')\), respectively. For example, if he dies during the year \((2, 3)\), he will have lived one complete year \((1, 2)\) and a fraction \(a_2'\) of the year \((2, 3)\), therefore he will have lived a total of \(1+a_2'\) years. The probability of dying at any time during the interval \((1, 5)\) is \(1 - p_1p_2p_3p_4\), and the length of the interval is 5-1 = 4 years, therefore the formula for the fraction \(a_1\) is

\[
a_1 = \frac{q_1a_1' + p_1q_1(1+a_2') + p_1p_2q_3(2+a_3') + p_1p_2p_3q_4(3+a_4')}{4(1 - p_1p_2p_3p_4)}.
\]
Table 3. Frequency distribution of deaths by interval of days lived in the last year of life for selected ages, total population, California, 1960

<table>
<thead>
<tr>
<th>Interval (in completed days)</th>
<th>Age at Death (in completed years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0-14</td>
<td>6,091</td>
</tr>
<tr>
<td>15-28</td>
<td>305</td>
</tr>
<tr>
<td>29-42</td>
<td>228</td>
</tr>
<tr>
<td>43-56</td>
<td>247</td>
</tr>
<tr>
<td>57-70</td>
<td>233</td>
</tr>
<tr>
<td>71-84</td>
<td>190</td>
</tr>
<tr>
<td>85-98</td>
<td>152</td>
</tr>
<tr>
<td>99-112</td>
<td>153</td>
</tr>
<tr>
<td>113-126</td>
<td>152</td>
</tr>
<tr>
<td>127-140</td>
<td>114</td>
</tr>
<tr>
<td>141-154</td>
<td>89</td>
</tr>
<tr>
<td>155-168</td>
<td>91</td>
</tr>
<tr>
<td>169-182</td>
<td>61</td>
</tr>
<tr>
<td>183-196</td>
<td>55</td>
</tr>
<tr>
<td>197-210</td>
<td>55</td>
</tr>
<tr>
<td>211-224</td>
<td>54</td>
</tr>
<tr>
<td>225-238</td>
<td>47</td>
</tr>
<tr>
<td>239-252</td>
<td>50</td>
</tr>
<tr>
<td>253-266</td>
<td>35</td>
</tr>
<tr>
<td>267-280</td>
<td>41</td>
</tr>
<tr>
<td>281-294</td>
<td>36</td>
</tr>
<tr>
<td>295-308</td>
<td>31</td>
</tr>
<tr>
<td>309-322</td>
<td>24</td>
</tr>
<tr>
<td>323-336</td>
<td>41</td>
</tr>
<tr>
<td>337-350</td>
<td>28</td>
</tr>
<tr>
<td>351-364</td>
<td>21</td>
</tr>
<tr>
<td>Total Deaths</td>
<td>8,624</td>
</tr>
</tbody>
</table>
x²  | 97,299.5**| 68.4**| 29.4| 22.9| 15.8| 26.6| 27.4| 21.0| 13.2| 40.9* |

* Significant at the 5 percent level
** Significant at the 1 percent level
Table 4. Computation of the fraction $a_0$ based on infant deaths, United States total population, 1970

<table>
<thead>
<tr>
<th>Age interval at death</th>
<th>Average point in interval (in days)</th>
<th>Number of deaths in interval*</th>
<th>Number of days lived (2) x (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;1 hour</td>
<td>0.02</td>
<td>6,485</td>
<td>129.7</td>
</tr>
<tr>
<td>1-24 hours</td>
<td>0.5</td>
<td>26,425</td>
<td>13,212.5</td>
</tr>
<tr>
<td>1-2 days</td>
<td>1.5</td>
<td>7,944</td>
<td>11,916.0</td>
</tr>
<tr>
<td>2-3 days</td>
<td>2.5</td>
<td>4,761</td>
<td>11,902.5</td>
</tr>
<tr>
<td>3-4</td>
<td>3.5</td>
<td>2,163</td>
<td>7,570.5</td>
</tr>
<tr>
<td>4-5</td>
<td>4.5</td>
<td>1,346</td>
<td>6,057.0</td>
</tr>
<tr>
<td>5-6</td>
<td>5.5</td>
<td>984</td>
<td>5,412.0</td>
</tr>
<tr>
<td>6-7</td>
<td>6.5</td>
<td>713</td>
<td>4,634.5</td>
</tr>
<tr>
<td>7-14</td>
<td>10.0</td>
<td>2,722</td>
<td>27,220.0</td>
</tr>
<tr>
<td>14-21</td>
<td>17.0</td>
<td>1,461</td>
<td>24,837.0</td>
</tr>
<tr>
<td>21-28</td>
<td>24.0</td>
<td>1,275</td>
<td>30,600.0</td>
</tr>
<tr>
<td>28-60</td>
<td>42.0</td>
<td>4,662</td>
<td>195,804.0</td>
</tr>
<tr>
<td>2-3 mos.</td>
<td>73.0</td>
<td>3,561</td>
<td>259,953.0</td>
</tr>
<tr>
<td>3-4</td>
<td>103.0</td>
<td>2,586</td>
<td>266,358.0</td>
</tr>
<tr>
<td>4-5</td>
<td>134.0</td>
<td>1,866</td>
<td>250,044.0</td>
</tr>
<tr>
<td>5-6</td>
<td>164.0</td>
<td>1,379</td>
<td>226,156.0</td>
</tr>
<tr>
<td>6-7</td>
<td>195.0</td>
<td>1,065</td>
<td>207,675.0</td>
</tr>
<tr>
<td>7-8</td>
<td>225.0</td>
<td>874</td>
<td>196,650.0</td>
</tr>
<tr>
<td>8-9</td>
<td>256.0</td>
<td>678</td>
<td>173,568.0</td>
</tr>
<tr>
<td>9-10</td>
<td>287.0</td>
<td>597</td>
<td>171,339.0</td>
</tr>
<tr>
<td>10-11</td>
<td>318.0</td>
<td>565</td>
<td>179,670.0</td>
</tr>
<tr>
<td>11-12</td>
<td>349.0</td>
<td>555</td>
<td>193,695.0</td>
</tr>
</tbody>
</table>

| Total                 | 74,667                             | 2,464,403.7                  |


\[
a_0 = \frac{2,464,403.7}{365 \times 74,667} = .09
\]
Using the established values of $a'_1$, $a'_2$, $a'_3$, and $a'_4$, we have

$$a_1 = \frac{.43q_1 + 1.45p_1q_2 + 2.47p_1p_2q_3 + 3.49p_1p_2p_3q_4}{4(1 - p_1p_2p_3p_4)} \quad (3.2)$$

For a given country, the given probabilities, $q_1$, $q_2$, $q_3$ and $q_4$ can be determined. Therefore, the fraction $a_1$ for the interval (1, 5) may be computed from formula (3.2). The computation of $a_1$ for California population, 1970, is demonstrated in Table 5.

**Table 5.** Computation of the fraction $a_1$ for age interval (1, 5) based on California mortality data, 1970

<table>
<thead>
<tr>
<th>Year of Age</th>
<th>Conditional Probability of Dying in year $(x, x+1)$ given alive at age 1</th>
<th>Expected length of time lived in interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Length of time lived</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(1, 5)$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1-2</td>
<td>$q_1 = .00113$</td>
<td>.43</td>
</tr>
<tr>
<td>2-3</td>
<td>$p_1q_2 = (.99887)(.00086) = .000859$</td>
<td>1.45</td>
</tr>
<tr>
<td>3-4</td>
<td>$p_1p_2q_3 = (.99887)(.99914)(.00073) = .000729$</td>
<td>2.47</td>
</tr>
<tr>
<td>4-5</td>
<td>$p_1p_2p_3q_4 = (.99887)(.99914)(.99927)(.00052) = .000519$</td>
<td>3.49</td>
</tr>
<tr>
<td>Total</td>
<td>$1 - p_1p_2p_3p_4 = .003236$</td>
<td></td>
</tr>
</tbody>
</table>

$$a_1 = \frac{.00534}{4 \times .003236} = .41$$
From age 5 to the last interval in the life table, the length of each age interval is 5 years and the fraction of last year of life for each year is \( a_x' = 1/2 \). The formula for the fraction \( a_i \) for interval \((x_i, x_{i+5})\) can be simplified somewhat. For age interval \((5, 10)\) for example, we have

\[
a_5 = \frac{.5q_5 + (1+.5)p_5q_6 + (2+.5)p_5p_6q_7 + (3+.5)p_5p_6p_7q_8 + (4+.5)p_5p_6p_7p_8q_9}{5(1 - p_5p_6p_7p_8p_9)}
\]

\[
= \frac{p_5q_5 + 2p_5p_6q_7 + 3p_5p_6p_7q_8 + 4p_5p_6p_7p_8q_9}{5(1 - p_5p_6p_7p_8p_9)} + .1
\]  

since

\[
q_5 + p_5q_6 + p_5p_6q_7 + p_5p_6p_7q_8 + p_5p_6p_7p_8q_9 = 1 - p_5p_6p_7p_8p_9.
\]  

The values of the fraction \( a_i \) for the abridged life table have been computed from formulas (3.2) and (3.3) for selected countries for which the required information is available, and are listed in Appendix V. These values of \( a_i \) can be used directly in constructing life tables for the respective countries.

Remark 4. Formulas (3.2) and (3.3) show that \( a_i \) does not depend on the absolute values of \( q_x \) or \( p_x \) but rather on the trend of mortality within the interval. For example, if \( q_5 > q_6 > q_7 > q_8 > q_9 \), then \( a_5 < 1/2 \), regardless of the absolute values of these \( q_x \)’s.

Remark 5. The probabilities \( q_x \) and \( p_x \) are computed from the mortality data of a population in question, the value of \( a_i \) represents the mortality trend in each interval prevailing in the population. Since the mortality trend does not vary much over time (although death rates do), the \( a_i \) values may be regarded as constant and may be used for the construction of abridged life tables of the subsequent years of the population.
The invariant property of \( a_i \) not only holds over time, but is also true for countries with similar mortality patterns. Table 8 shows a remarkable agreement of the five sets of \( a_i \) values. For countries with a similar mortality pattern, the same set of \( a_i \) values may be used.

Remark 6: The assumption that \( a' = 1/2 \) for each year of age within an interval \((x_i, x_{i+1})\) does not necessarily imply that \( a_i = 1/2 \) for the entire interval. As formula (3.3) shows, the value of the fraction \( a_i \) depends on the mortality pattern over an entire interval and not on the mortality rate for any single year. When the mortality rate increases with age in an interval, the fraction \( a_i > 1/2 \); then the reverse pattern prevails, \( a_i < 1/2 \). Consider, for example, the age intervals (5, 10) and (10, 15) in 1970 California population. Although \( a'_x = 1/2 \) for each age in the two intervals, \( a_2 = .44 \) for interval (5, 10) and \( a_3 = .54 \) for interval (10, 15) due to the changing mortality pattern, as shown in Table 7.
Table 6

Fraction of last age interval of life, $a_i$, for selected populations

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.12</td>
<td>.09</td>
<td>.16*</td>
<td>.09</td>
<td>.09</td>
</tr>
<tr>
<td>1-5</td>
<td>.37</td>
<td>.41</td>
<td>.38</td>
<td>.38</td>
<td>.40</td>
</tr>
<tr>
<td>5-10</td>
<td>.47</td>
<td>.44</td>
<td>.46</td>
<td>.49</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.51</td>
<td>.54</td>
<td>.54</td>
<td>.52</td>
<td>.55</td>
</tr>
<tr>
<td>15-20</td>
<td>.58</td>
<td>.59</td>
<td>.56</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.48</td>
<td>.49</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.52</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.52</td>
<td>.53</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.51</td>
<td>.53</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
<td>.51</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.48</td>
<td>.50</td>
<td>.49</td>
<td>.47</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.45</td>
<td>.46</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>90-95</td>
<td>.40</td>
<td>.41</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*A large $a_0$ value for the France 1969 population is due to the fact that infants who die before 3 days old are not recorded. Age at death of these infants are not included in the calculation of $a_0$.\*
Table 7
Computation of $a_i$ for age intervals (5, 10) and (10, 15)
based on California population, 1970

<table>
<thead>
<tr>
<th>Age interval</th>
<th>Fraction of the last year of life</th>
<th>Proportion dying in age interval</th>
<th>Fraction of last age interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ to $x+1$</td>
<td>$a'_x$</td>
<td>$q_x$</td>
<td>$a_i$</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td></td>
</tr>
<tr>
<td>5-6</td>
<td>.50</td>
<td>.00049</td>
<td></td>
</tr>
<tr>
<td>6-7</td>
<td>.50</td>
<td>.00045</td>
<td></td>
</tr>
<tr>
<td>7-8</td>
<td>.50</td>
<td>.00034</td>
<td></td>
</tr>
<tr>
<td>8-9</td>
<td>.50</td>
<td>.00031</td>
<td></td>
</tr>
<tr>
<td>9-10</td>
<td>.50</td>
<td>.00030</td>
<td></td>
</tr>
<tr>
<td>10-11</td>
<td>.50</td>
<td>.00031</td>
<td></td>
</tr>
<tr>
<td>11-12</td>
<td>.50</td>
<td>.00033</td>
<td></td>
</tr>
<tr>
<td>12-13</td>
<td>.50</td>
<td>.00035</td>
<td>.54</td>
</tr>
<tr>
<td>13-14</td>
<td>.50</td>
<td>.00041</td>
<td></td>
</tr>
<tr>
<td>14-15</td>
<td>.50</td>
<td>.00048</td>
<td></td>
</tr>
</tbody>
</table>
4. Significant Historical Contributions to the Construction of Abridged Life Tables

The history of life table construction reflects increasing refinement of the method. For instance, although the earliest tables (see Introduction to this chapter) were based solely on recorded deaths, Milne's table of 1815 took into account population figures as well. In 1839 the English Life Tables were constructed using only registered births and deaths since, due to the influence of William Farr, census figures were found to be unreliable. Other significant contributions and refinements followed, in particular those of Moore, Day, Wickens, Pell, King, Derksen, Greville, Reed-Merrell, Wiesler, Keyfitz, and Sirken. We shall briefly discuss some of these methods below.

4.1. King's Method. This method was introduced by George King in the construction of the Seventh English Life Table at the turn of the century and has been used by many English-speaking countries for fifty years or so. Using this method, data are arranged in five-year age groups. Population figures and the number of deaths are calculated for the central year of age (pivotal age) of each age group by a graduation process, yielding the values of $q_x$ for the pivotal age. Using the complement of $q_x$ at pivotal ages and finite difference formulas, the number of survivors ($Q_x$) are obtained. T. N. E. Greville adapted this method for the 1939-41 United States Life Tables.

4.2. Reed-Merrell Method. In the search for a relation between the probability $q_1$ and the mortality rate, $M_1$, Lowell J. Reed and Margaret Merrell studied extensively some thirty-three tables in J. W. Glover's 1910 series of United States Life Tables. Their findings were published in 1939 showing that the following equation describes
satisfactorily the entire range of observations in Clover’s tables:

\[ q_i = 1 - e^{-n_i M_i - 0.008 n_i M_i^2} \]

Many formulas are also given to determine the \( \ell_i \) column from the number of survivors \( \ell_i \) in the life table.

4.3. Greville’s Method. T.N.E. Greville used a mathematical approach to derive a relation between \( q_i \) and \( M_i \). He started with the equation

\[ M_i = \frac{d}{dx} \log L_i \]

After integrating both sides of the equation, thus yielding \( L_i \), and applying the Euler-Maclaurin summation formula, Greville was able to express \( T_i \) in terms of a series of exponential functions of \( M_i \). He then used quite skilful mathematical manipulations, and arrived at the formula:

\[ q_i = \frac{M_i}{1/n_i + M_i[1/2 + n_i/12(M_i - \log c)]]} \]

where the constant \( c \) is the constant in the Gompertz’ law of mortality:

\[ \mu_x = be^{bx} \]

Greville also suggested a number of formulas to compute the life table population \( L_i \).
4.4. Wiesler's Method. This method, proposed in "Une méthodes simple pour la construction de tables de mortalité abrégées," World Population Conference, 1954, Volume IV, United Nations, in essence uses age specific death rates $M_{i}$ as the probability of dying $q_{i}$. For an age interval $(x_{i}, x_{i}+n_{i})$, let $D_{i}$ be the number of deaths during a calendar year and $P_{i}$ be the total of living people in the age group $(x_{i}, x_{i}+n_{i})$. Then Wiesler suggests that

$$\hat{p}_{i} = 1 - \frac{D_{i}}{P_{i}} \quad \text{or} \quad \hat{q}_{i} = \frac{D_{i}}{P_{i}},$$

and $l_{1}, l_{5}, l_{10}$, etc., are computed successively from

$$l_{1} = \hat{L}_{0} = 0,$$
$$l_{5} = l_{1}(p_{1-4})^{t_{1-4}},$$
$$l_{10} = l_{5}(p_{5-9})^{t_{5-9}}, \ldots,$$

where $t_{1-4}, t_{5-9}, \ldots$ are assumed to be the same for all mortality tables.

The expectation of life at $x_{\alpha}$ is computed from

$$\hat{e}_{\alpha} = \frac{1}{2} + \frac{l_{\alpha+1} + l_{\alpha+2} + \ldots}{l_{\alpha}}.$$

4.5. Sirken's Method. Monroe Sirken distinguishes the age specific death rate $M_{i}$ from a current population:

$$M_{i} = \frac{D_{i}}{P_{i}},$$

and the rate $m_{i}$ defined in the life table quantities:

$$m_{i} = \frac{d_{i}}{L_{i}}.$$

Using the observed death rate $M_{i}$, one derives $q_{i}$ from the equation
where the constant \( \alpha \) is assumed to be the same as in a standard table.

Using \( q_i \), one completes the columns \( l_i \) and \( d_i \). To compute \( L_i \), Sirken considers another equation

\[
q_i = \frac{n_i M_i}{1 + \alpha_i M_i} \tag{A}
\]

Substituting \( q_i = \frac{d_i}{\hat{\lambda}_i} \) and \( m_i = \frac{d_i}{L_i} \) in (B) yields

\[
\frac{d_i}{\hat{\lambda}_i} = \frac{n_i d_i}{L_i} \tag{B}
\]

Solving the last equation for \( L_i \) one gets

\[
L_i = n_i \hat{\lambda}_i - a_i d_i
\]

where the constant \( a_i \) is assumed to be the same as in a standard table but is different from \( q_i \).

4.5. Keyfitz's Method. This is an iterative procedure using the basic relationship between the probability \( q_i \) and the age-specific mortality rate \( m_i \) or \( M_i \)

\[
q_i = \frac{n_i M_i}{1 + (n_i - a_i) M_i} \tag{A}
\]

where \( n_i \) is the number of years lived in the age interval \( (x_i, x_i + n_i) \) by an individual who dies in the interval. In addition to \( n_i \), Keyfitz introduces a quantity \( A_i \), the average number of years already lived within the interval \( (x_i, x_i + n_i) \) by a stationary population aged \( (x_i, x_i + n_i) \).

Taking \( n_i / 2 \) on the first cycle to obtain first approximation of \( q_i \) using formula (A), then using
and other formulas to arrive at a second approximation of \( q_i \). After each iteration, a life table is constructed and the age-specific mortality rate is compared with those observed, and an adjustment made for the next iteration. The iterative process continues until the life table age specific rates agree with the corresponding observed rates.
5. Cohort (Generation) Life Table

It has been pointed out in Section 1 of this chapter that a cohort life table describes the actual mortality experience of a particular group of individuals (the cohort) from birth to the death of the last member of the group. The subject involved need not be human beings. Cohort life tables for various animal populations have appeared frequently in the literature. In fact, the cohort life table has been widely used for years in studies of animal populations, in biological control, in ecology, and in population dynamics. Cohort life tables have been constructed for inanimate objects as well.

For simplicity of formulas, the length of age interval is assumed to be constant and denoted by \( n \). The basic variables involved in a cohort life table are the number of survivors (\( \ell_x \)) at each age \( x \), the number of deaths (\( d_x \)) within each age interval \((x, x+n)\). The unit of \( x \) depends on the problem in question. In any case, the numbers \( \ell_x \) and \( d_x \) satisfy the obvious relationship

\[
\ell_x - \ell_{x+n} = d_x
\]  
(5.1)

or

\[
\ell_{x+n} = \ell_x - d_x
\]  
(5.2)

The number of survivors at age \( x+n \) is equal to the number alive at the beginning of the interval \((x, x+n)\) minus those who died during the interval.

The probability \( q_x \) for each interval is estimated by dividing \( d_x \) by \( \ell_x \),

\[
\hat{q}_x = \frac{d_x}{\ell_x}
\]  
(5.3)

When \( \ell_x \), \( d_x \), and \( \hat{q}_x \) are determined, the remaining part of the life table can be completed in exactly the same way as in the current life table in Section 1 and 2.

Assuming \( n_x = 1/2 \), we have
\[ L_x = n \ell_{x+n} + (1 - \frac{1}{2})n d_x = \frac{n}{2} (\ell_x + \ell_{x+n}) \]  

(5.4)

In a cohort life table, observations are usually made throughout the life span of subjects under study. Therefore, the values \( \ell_x \), \( d_x \), \( L_x \) and formulas from (5.1) to (5.5) are all applicable to the last age interval (\( \ell_x \) are over). The quantity \( T_x \) as before is equal to the sum of \( \ell_x \), i.e.,

\[ T_x = \ell_x + \ldots + \ell_w \]  

(5.5)

where the symbol \( w \) indicates the beginning of the last age interval. Finally, the expectation of life \( \ell_x \) is given by

\[ \hat{\ell}_x = \frac{T_x}{\ell_x}, \quad x = 0, 1, \ldots, w \]  

(5.6)

An example of a life table for adult Drosophila Melanogaster is presented in Table 10 for illustration [Miller and Thomas]. A group of \( l_0 = 270 \) male fruit flies were followed from the time they became adults. The number of survivors at each five day period and the number of deaths occurring within each age interval of five days are recorded in columns (2) and (3) respectively. Dividing \( d_x \) by \( \ell_x \) for each age interval gives the probability of dying \( q_x \) in column (4). Using relations (5.4), (5.5), and (5.6), we computed the quantities \( \ell_x \), \( T_x \), and \( \hat{\ell}_x \) for each age, and recorded the numerical values in columns (5), (6), and (7), respectively assuming \( a_x = .05 \) over all intervals. A similar table has been constructed for female adult Drosophila. Comparison between the two sexes with respect to the expectation of life, the survival probability, or the probability of death, can easily be made with the aid of the corresponding standard deviations (c.f., Chapter 6).
Table 8

Life Table of Adult Male Drosophila Melanogaster

<table>
<thead>
<tr>
<th>Age Interval (Days)</th>
<th>Number living at age x</th>
<th>Number dying in (x, x+n)</th>
<th>Probability of dying in (x, x+n)</th>
<th>Days lived in (x, x+n)</th>
<th>Days lived beyond age x</th>
<th>Observed Expectation of life at age x</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, x+n)</td>
<td>( l_x )</td>
<td>( d_x )</td>
<td>( \hat{q}_x )</td>
<td>( L_x )</td>
<td>( T_x )</td>
<td>( \hat{e}_x )</td>
</tr>
<tr>
<td>0-5</td>
<td>270</td>
<td>2</td>
<td>.00741</td>
<td>1345</td>
<td>11660</td>
<td>43.2</td>
</tr>
<tr>
<td>5-10</td>
<td>268</td>
<td>4</td>
<td>.01493</td>
<td>1330</td>
<td>10315</td>
<td>38.5</td>
</tr>
<tr>
<td>10-15</td>
<td>264</td>
<td>3</td>
<td>.01136</td>
<td>1312</td>
<td>8985</td>
<td>34.0</td>
</tr>
<tr>
<td>15-20</td>
<td>261</td>
<td>7</td>
<td>.02682</td>
<td>1288</td>
<td>7673</td>
<td>29.4</td>
</tr>
<tr>
<td>20-25</td>
<td>254</td>
<td>3</td>
<td>.01181</td>
<td>1262</td>
<td>6385</td>
<td>25.1</td>
</tr>
<tr>
<td>25-30</td>
<td>251</td>
<td>3</td>
<td>.01195</td>
<td>1248</td>
<td>5123</td>
<td>20.4</td>
</tr>
<tr>
<td>30-35</td>
<td>248</td>
<td>16</td>
<td>.06452</td>
<td>1200</td>
<td>3875</td>
<td>15.6</td>
</tr>
<tr>
<td>35-40</td>
<td>232</td>
<td>66</td>
<td>.28448</td>
<td>995</td>
<td>2675</td>
<td>11.5</td>
</tr>
<tr>
<td>40-45</td>
<td>166</td>
<td>36</td>
<td>.21687</td>
<td>740</td>
<td>1680</td>
<td>10.1</td>
</tr>
<tr>
<td>45-50</td>
<td>150</td>
<td>54</td>
<td>.41538</td>
<td>515</td>
<td>940</td>
<td>7.2</td>
</tr>
<tr>
<td>50-55</td>
<td>76</td>
<td>42</td>
<td>.55263</td>
<td>275</td>
<td>425</td>
<td>5.6</td>
</tr>
<tr>
<td>55-60</td>
<td>34</td>
<td>21</td>
<td>.61765</td>
<td>118</td>
<td>150</td>
<td>4.4</td>
</tr>
<tr>
<td>60+</td>
<td>13</td>
<td>13</td>
<td>1.00000</td>
<td>32</td>
<td>32</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Table 9
Life Table of Adult Female Drosophila Melanogaster

<table>
<thead>
<tr>
<th>Age Interval (Days)</th>
<th>Number living at age x</th>
<th>Number dying in (x, x+n)</th>
<th>Probability of dying in (x, x+n)</th>
<th>Days lived in (x, x+n)</th>
<th>Days lived beyond x</th>
<th>Observed Expectation of life at age x</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, x+n)</td>
<td>( \ell_x )</td>
<td>( d_x )</td>
<td>( q_x )</td>
<td>( L_x )</td>
<td>( T_x )</td>
<td>( \hat{e}_x )</td>
</tr>
<tr>
<td>0-5</td>
<td>275</td>
<td>4</td>
<td>.01455</td>
<td>1365</td>
<td>10303</td>
<td>37.5</td>
</tr>
<tr>
<td>5-10</td>
<td>271</td>
<td>7</td>
<td>.02583</td>
<td>1338</td>
<td>8938</td>
<td>33.0</td>
</tr>
<tr>
<td>10-15</td>
<td>264</td>
<td>3</td>
<td>.01136</td>
<td>1312</td>
<td>7600</td>
<td>28.8</td>
</tr>
<tr>
<td>15-20</td>
<td>261</td>
<td>7</td>
<td>.02682</td>
<td>1288</td>
<td>6288</td>
<td>24.1</td>
</tr>
<tr>
<td>20-25</td>
<td>254</td>
<td>13</td>
<td>.05118</td>
<td>1238</td>
<td>5000</td>
<td>19.7</td>
</tr>
<tr>
<td>25-30</td>
<td>241</td>
<td>22</td>
<td>.09129</td>
<td>1150</td>
<td>3762</td>
<td>15.6</td>
</tr>
<tr>
<td>30-35</td>
<td>219</td>
<td>31</td>
<td>.14155</td>
<td>1018</td>
<td>2612</td>
<td>11.9</td>
</tr>
<tr>
<td>35-40</td>
<td>188</td>
<td>68</td>
<td>.36170</td>
<td>770</td>
<td>1594</td>
<td>8.5</td>
</tr>
<tr>
<td>40-45</td>
<td>120</td>
<td>51</td>
<td>.42500</td>
<td>472</td>
<td>824</td>
<td>6.9</td>
</tr>
<tr>
<td>45-50</td>
<td>69</td>
<td>38</td>
<td>.55072</td>
<td>250</td>
<td>352</td>
<td>5.1</td>
</tr>
<tr>
<td>50-55</td>
<td>31</td>
<td>26</td>
<td>.83871</td>
<td>90</td>
<td>102</td>
<td>3.3</td>
</tr>
<tr>
<td>55 +</td>
<td>5</td>
<td>5</td>
<td>1.00000</td>
<td>12</td>
<td>12</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Footnotes

1/ During the early development of the concept of expectation of life, a curtate expectation of life, defined as

\[ e_x = \frac{\ell_{x+1} + \ell_{x+2} + \ldots}{\ell_x} \]

was first introduced. This expectation considers only the completed years of life lived by survivors, whereas the complete expectation of life takes into account also the fractional years of life lived by those who die in any year. Under the assumption that each person who dies during any year of age lives half of the year on the average, the complete expectation of life is given by

\[ e_x = e_x + \frac{1}{2} \]

Since the curtate expectation is no longer in use, in this book the symbol \( e_x \) is used to denote the true expectation of life at age \( x \).
CHAPTER 6

STATISTICAL INFERENCE REGARDING LIFE TABLE FUNCTIONS

1. Introduction

Each figure in a life table as described in the preceding chapters is an estimate of the corresponding unknown true value. Statistical inference regarding these unknown values may be made on the basis of the observed quantities. An essential element required in making statistical inference, as indicated in Chapter 3, is the standard error of the estimate. The purpose of this chapter is (1) to derive formulas for the sample variances (or their square roots, standard error) of the life table functions, and (2) to demonstrate with numerical values how to construct confidence intervals and how to test statistical hypotheses. Specifically, inference will be made about three categories of parameters: (i) $q_i$, the probability of dying in an age interval $(x_i, x_{i+1})$; (ii) $p_{ij}$, the survival probability from age $x_i$ to $x_j$; and (iii) $e_{x}$, the expectation of life at age $x$, for $\alpha = 0, 1, ..., w$.

2. The Probability of Dying $q_i$ and the Probability of Surviving $p_i$

The probability of dying and the probability of surviving an age interval are complementary to one another; therefore their estimates have the same sample variance. Denoting the sample variances by $S^2_{q_i}$ and $S^2_{p_i}$, respectively, we have

$$S^2_{q_i} = S^2_{p_i}.\quad (2.1)$$

In a current life table, the estimate $\hat{q}_i$ is derived from the corresponding mortality information, in terms of which the sample variance of $\hat{q}_i$ should be expressed. We have found in Chapter 3, equation (3.5) that

$$S^2_{\hat{q}_i} = \frac{1}{D_i} q_i^2 (1-q_i),\quad (2.2)$$
and the 95% confidence interval for the probability $q_i$:

$$\Pr\{ \hat{q}_i - 1.96 \frac{S_{q_i}}{q_i} < q_i < \hat{q}_i + 1.96 \frac{S_{q_i}}{q_i} \} = .95 . \quad (2.3)$$

For a given problem, $q_i$ and $S_{q_i}$ can be determined, and the two limits, $\hat{q}_i - 1.96 \frac{S_{q_i}}{q_i}$ and $\hat{q}_i + 1.96 \frac{S_{q_i}}{q_i}$, can be found. These limits are called the confidence limits, and the interval extending from the lower limit $\hat{q}_i - 1.96 \frac{S_{q_i}}{q_i}$ to the upper limit $\hat{q}_i + 1.96 \frac{S_{q_i}}{q_i}$ is the 95% confidence interval.

As an example, consider the probability of dying in the first year of life, $q_0$. In the 1970 California experience, the estimate $\hat{q}_0 = .01801$, the number of deaths, $D_0 = 6234$, and hence the standard error of $\hat{q}_0$ is:

$$S_{\hat{q}_0} = \sqrt{\frac{\hat{q}_0(1-\hat{q}_0)}{D_0}}$$

$$= \sqrt{\frac{(.01801)(1-.01801)}{6234}}$$

$$= .000226 .$$

Substituting these values in (2.3) yields the 95% confidence limits for the probability $q_0$:

$$\hat{q}_0 - 1.96 \frac{S_{\hat{q}_0}}{q_0} = .01801 - 1.96(.000226) = .01757$$

$$\hat{q}_0 + 1.96 \frac{S_{\hat{q}_0}}{q_0} = .01801 + 1.96(.000226) = .01845 .$$

Thus we conclude with a 95% confidence that, if the California 1970 mortality experience prevails in a population, the probability that a newborn will not survive to the first birthday is between .01757 and .01845.
The logic of the preceding statement needs some explanation. Formula (2.3) indicates that, before information is gathered, the chances are 95 out of 100 that the interval \( (q_i - 1.96 S_{q_i}, q_i + 1.96 S_{q_i}) \) to be determined will contain the unknown quantity \( q_i \). After the information is gathered, and the numerical values of the limits (.01757 and .01845) are obtained, we certainly have confidence in the statement that the quantity \( q_0 \) is between .01757 and .01845; a measure of this confidence is the value of the probability .95. This measure of confidence (.95, in this case) is called the confidence coefficient. The essential point to be recognized is that a probability is a measure of likelihood of occurrence of an event (death, for example) before the event takes place, whereas a confidence coefficient is a measure of confidence one has in a statement about an unknown quantity after the corresponding event has occurred.

A second use of the sample variance of the estimate \( \hat{q}_i \) is testing a hypothesis concerning either the probability of dying in one age interval or the comparison of two or more probabilities. Suppose we want to know if the force of mortality has decreased over the past decade so that a new born in 1970 has a better chance of surviving the first year of life than that in 1960. Here we are testing the hypothesis that \( q_0(1970) \) is the same as \( q_0(1960) \) against the alternative hypothesis that \( q_0(1970) \) is smaller than \( q_0(1960) \). The statistics for the test is

\[
Z = \frac{\hat{q}_0(1960) - \hat{q}_0(1970)}{S.E.[\hat{q}_0(1960) - \hat{q}_0(1970)]}
\]

(2.4)

where the standard error of the difference is given by

\[
S.E.[\hat{q}_0(1960) - \hat{q}_0(1970)] = \sqrt{\frac{\hat{q}_0^2(1960)[1-\hat{q}_0(1960)]}{D_0(1960)}} + \frac{\hat{q}_0^2(1970)[1-\hat{q}_0(1970)]}{D_0(1970)}
\]

(2.5)
Using California experience again, we have the required information given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>1960</th>
<th>1970</th>
<th>( \hat{q}_0(1960) - \hat{q}_0(1970) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{q}_0 )</td>
<td>0.02378</td>
<td>0.01801</td>
<td>0.00577</td>
</tr>
<tr>
<td>( D_0 )</td>
<td>8663</td>
<td>6234</td>
<td></td>
</tr>
<tr>
<td>( S^2 )</td>
<td>6.3724 \times 10^{-8}</td>
<td>5.109 \times 10^{-8}</td>
<td>11.481 \times 10^{-8}</td>
</tr>
<tr>
<td>S.E.</td>
<td>2.524 \times 10^{-4}</td>
<td>2.260 \times 10^{-4}</td>
<td>3.388 \times 10^{-4}</td>
</tr>
</tbody>
</table>

From Table 1 we compute the statistic

\[ Z = \frac{0.00577}{3.388 \times 10^{-4}} = 17.03 , \]

which is significantly greater than the 99th percentile in the standard normal distribution. We conclude that a newborn in California in 1970 has a smaller probability of dying in the first year of life than that in 1960.

Remark 1. In a cohort (generation) life table both the number living \((L_i)\) at age \(x_i\) and the number of deaths \((d_i)\) occurring in the interval \((x_i, x_{i+1})\) are directly observed. The probabilities are estimated by

\[ \hat{q}_i = \frac{d_i}{L_i} \quad \text{and} \quad \hat{p}_i = 1 - \frac{d_i}{L_i} . \quad (2.6) \]

Here we have a binomial situation, so that the variance of \(\hat{q}_i (\hat{p}_i)\) is given by

\[ S^2_{\hat{q}_i} = \frac{1}{L_i} \frac{\hat{q}_i (1-\hat{q}_i)}{\hat{p}_i} = S^2_{\hat{p}_i} . \quad (2.7) \]
3. The Survival Probability, \( p_{ij} \)

The probability that a person of age \( x_i \) will survive to age \( x_j \) is an important quantity in the survival analysis. It provides an investigator with critical information that he seeks in his study. This probability can be obtained directly from the life table. Since the survival of a person from age \( x_i \) to \( x_j \) means the survival of every single intermediate age interval, the probability \( p_{ij} \) is given by the equation

\[
p_{ij} = p_i \cdot p_{i+1} \cdots p_{j-1}
\]

or

\[
p_{ij} = (1-q_i)(1-q_{i+1}) \cdots (1-q_{j-1}) .
\]

A case of particular interest is when \( x_i = 0 \). Here we have the probability of surviving from age 0 to a specified age \( x_j \)

\[
P_{0j} = p_0 \cdot p_1 \cdots p_{j-1}
\]

\[
= (1-q_0)(1-q_1) \cdots (1-q_{j-1}) .
\]

To obtain the estimate of the survival probability, we only need to substitute the estimates of \( \hat{q}_i \) in the formulas (3.2) and (3.3). When the information is taken from a life table, computations can be simplified. For example,

\[
\hat{P}_{0j} = \hat{p}_0 \cdot \hat{p}_1 \cdots \hat{p}_{j-1}
\]

\[
= \frac{\hat{\ell}_1}{\hat{\ell}_0} \cdot \frac{\hat{\ell}_2}{\hat{\ell}_1} \cdots \frac{\hat{\ell}_j}{\hat{\ell}_{j-1}} = \frac{\hat{\ell}_j}{\hat{\ell}_0} ,
\]

similarly

\[
\hat{p}_{ij} = \frac{\hat{\ell}_j}{\hat{\ell}_i} .
\]
In the current life table the individual estimates,
\[ \hat{p}_h = 1 - \hat{q}_h, \quad h = i, \ldots, j-1, \] (3.6)
are based on the corresponding age-specific death rates, the sample variance
of \( \hat{p}_{ij} \) should be expressed in terms of the sample variance of each \( \hat{q}_h \). Since the individual estimates \( \hat{q}_h \) are based on mortality information of separate age groups, they are statistically independent of one another.

Using a theorem on the variance of a product of independent random variables, the sample variance of \( \hat{p}_{ij} \) may be determined from the formula:
\[ \frac{s^2}{\hat{p}_{ij}} = \hat{p}_{ij}^{-2} \sum_{h=i}^{j-1} \hat{p}_h^{-2} s_h^2 \] (3.7)
with the sample variance of \( \hat{p}_h \) given in (2.2).

For the 1960 United States data and for the 1970 California data, the probability \( p_{oi} \) and the corresponding sample variances and standard errors have been computed. The numerical results are given in Table 3 and Table 4, respectively. The mean steps in the computation are as follows:

1. Record the number of deaths \( D_i \) occurring in each age interval in the population in Column 2, and the probability of dying in Column 3.
2. Use formula (2.2) to compute the sample variance of \( \hat{q}_i \) and enter it in Column 4.
3. Use formula (3.3) to compute the probability of surviving age interval \( (0, x_i) \hat{p}_{oi} \), and record it in Column 5. \( \hat{p}_{00} \) is 1 by definition.
4. Use formula (3.7) or
\[ \frac{s^2}{\hat{p}_{oi}} = p_{oi}^{-2} \left[ p_0^{-2} s_0^2 + p_1^{-2} s_1^2 + \ldots + p_{i-1}^{-2} s_{i-1}^2 \right] \]
to compute the variance of \( \hat{p}_{oi} \) and record it in Column 6.
Table 2

Abridged life table for total United States population, 1960

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Proportion Dying in Interval ( (x_i, x_{i+1}) ) ( q_i )</th>
<th>Number Living at Age ( x_i ) ( \lambda_i )</th>
<th>Number Dying in Interval ( (x_i, x_{i+1}) ) ( d_i )</th>
<th>Fraction of Last Age Interval of Life ( a_i )</th>
<th>No. Years Lived in Interval ( (x_i, x_{i+1}) ) ( L_i )</th>
<th>Total No. Years Lived Beyond Age ( x_i ) ( T_i )</th>
<th>Observed Expectation of Life at Age ( x_i ) ( e_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i ) to ( x_{i+1} )</td>
<td>( \hat{q}_i )</td>
<td>( \lambda_i )</td>
<td>( d_i )</td>
<td>( a_i )</td>
<td>( L_i )</td>
<td>( T_i )</td>
<td>( e_i )</td>
</tr>
<tr>
<td>0-1</td>
<td>.02623</td>
<td>100,000</td>
<td>2,623</td>
<td>.10</td>
<td>97,639</td>
<td>6,965,395</td>
<td>69.65</td>
</tr>
<tr>
<td>1-5</td>
<td>.00436</td>
<td>97,377</td>
<td>425</td>
<td>.39</td>
<td>388,471</td>
<td>6,867,756</td>
<td>70.53</td>
</tr>
<tr>
<td>5-10</td>
<td>.00245</td>
<td>96,952</td>
<td>238</td>
<td>.46</td>
<td>484,117</td>
<td>6,479,285</td>
<td>66.83</td>
</tr>
<tr>
<td>10-15</td>
<td>.00219</td>
<td>96,714</td>
<td>212</td>
<td>.54</td>
<td>483,082</td>
<td>5,995,168</td>
<td>61.99</td>
</tr>
<tr>
<td>15-20</td>
<td>.00458</td>
<td>96,502</td>
<td>442</td>
<td>.57</td>
<td>481,560</td>
<td>5,512,086</td>
<td>57.12</td>
</tr>
<tr>
<td>20-25</td>
<td>.00616</td>
<td>96,060</td>
<td>592</td>
<td>.49</td>
<td>478,790</td>
<td>4,030,526</td>
<td>52.37</td>
</tr>
<tr>
<td>25-30</td>
<td>.00652</td>
<td>95,468</td>
<td>622</td>
<td>.50</td>
<td>475,785</td>
<td>4,551,736</td>
<td>47.68</td>
</tr>
<tr>
<td>30-35</td>
<td>.00800</td>
<td>94,846</td>
<td>759</td>
<td>.52</td>
<td>472,408</td>
<td>4,075,951</td>
<td>42.97</td>
</tr>
<tr>
<td>35-40</td>
<td>.01159</td>
<td>94,087</td>
<td>1,090</td>
<td>.54</td>
<td>467,928</td>
<td>3,603,543</td>
<td>38.30</td>
</tr>
<tr>
<td>40-45</td>
<td>.01840</td>
<td>92,997</td>
<td>1,711</td>
<td>.54</td>
<td>461,050</td>
<td>3,135,615</td>
<td>33.72</td>
</tr>
<tr>
<td>45-50</td>
<td>.02902</td>
<td>91,286</td>
<td>2,649</td>
<td>.54</td>
<td>450,337</td>
<td>2,674,565</td>
<td>29.30</td>
</tr>
<tr>
<td>50-55</td>
<td>.04571</td>
<td>88,637</td>
<td>4,052</td>
<td>.53</td>
<td>433,663</td>
<td>2,224,228</td>
<td>25.09</td>
</tr>
<tr>
<td>55-60</td>
<td>.06577</td>
<td>84,585</td>
<td>5,563</td>
<td>.52</td>
<td>409,574</td>
<td>1,790,565</td>
<td>21.17</td>
</tr>
<tr>
<td>60-65</td>
<td>.10257</td>
<td>79,022</td>
<td>8,105</td>
<td>.52</td>
<td>375,658</td>
<td>1,380,991</td>
<td>17.48</td>
</tr>
<tr>
<td>65-70</td>
<td>.14763</td>
<td>70,917</td>
<td>10,469</td>
<td>.52</td>
<td>329,459</td>
<td>1,005,333</td>
<td>14.18</td>
</tr>
<tr>
<td>70-75</td>
<td>.21472</td>
<td>60,448</td>
<td>12,979</td>
<td>.51</td>
<td>270,441</td>
<td>675,874</td>
<td>11.18</td>
</tr>
<tr>
<td>75-80</td>
<td>.31280</td>
<td>47,469</td>
<td>14,848</td>
<td>.51</td>
<td>200,967</td>
<td>405,433</td>
<td>8.54</td>
</tr>
<tr>
<td>80-85</td>
<td>.46312</td>
<td>32,621</td>
<td>15,107</td>
<td>.48</td>
<td>123,827</td>
<td>204,466</td>
<td>6.27</td>
</tr>
<tr>
<td>85-90</td>
<td>.61437</td>
<td>17,514</td>
<td>10,760</td>
<td>.45</td>
<td>57,980</td>
<td>80,639</td>
<td>4.60</td>
</tr>
<tr>
<td>90-95</td>
<td>.78812</td>
<td>6,754</td>
<td>5,323</td>
<td>.41</td>
<td>18,067</td>
<td>22,659</td>
<td>3.35</td>
</tr>
<tr>
<td>95+</td>
<td>1.00000</td>
<td>1,431</td>
<td>1,431</td>
<td>4,592</td>
<td>4,592</td>
<td>3.21</td>
<td></td>
</tr>
</tbody>
</table>
### Table 3

Computation of the standard error of survival probability.

Total United States population, 1960.

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Number of Deaths in Interval $(x_i, x_{i+1})$</th>
<th>Probability of Dying in Interval $(x_i, x_{i+1})$</th>
<th>Sample Variance of $\hat{q}_i$</th>
<th>Probability of Surviving Interval $(0, x_i)$</th>
<th>Sample Variance of $\hat{p}_0i$</th>
<th>Standard Error of $\hat{p}_0i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_i, x_{i+1})$</td>
<td>$D_i$</td>
<td>$\hat{q}_i$</td>
<td>$10^8 x_s^2 \hat{q}_i$</td>
<td>$\hat{p}_0i$</td>
<td>$10^8 x_s^2 \hat{p}_0i$</td>
<td>$10^4 x_s^2 \hat{p}_0i$</td>
</tr>
<tr>
<td>0 - 1</td>
<td>110873.</td>
<td>0.026230</td>
<td>0.60426</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>1 - 5</td>
<td>17682.</td>
<td>0.004360</td>
<td>0.10703</td>
<td>0.973770</td>
<td>0.60426</td>
<td>0.77734</td>
</tr>
<tr>
<td>5 - 10</td>
<td>9163.</td>
<td>0.002450</td>
<td>0.06534</td>
<td>0.969520</td>
<td>0.70649</td>
<td>0.83695</td>
</tr>
<tr>
<td>10 - 15</td>
<td>7374.</td>
<td>0.002190</td>
<td>0.06489</td>
<td>0.967140</td>
<td>0.75848</td>
<td>0.87091</td>
</tr>
<tr>
<td>15 - 20</td>
<td>12185.</td>
<td>0.004580</td>
<td>0.17136</td>
<td>0.965020</td>
<td>0.81586</td>
<td>0.90325</td>
</tr>
<tr>
<td>20 - 25</td>
<td>13348.</td>
<td>0.006160</td>
<td>0.28252</td>
<td>0.960600</td>
<td>0.96799</td>
<td>0.98386</td>
</tr>
<tr>
<td>25 - 30</td>
<td>14214.</td>
<td>0.006520</td>
<td>0.29712</td>
<td>0.954680</td>
<td>1.21680</td>
<td>1.10308</td>
</tr>
<tr>
<td>30 - 35</td>
<td>19220.</td>
<td>0.008000</td>
<td>0.33066</td>
<td>0.948460</td>
<td>1.47180</td>
<td>1.21317</td>
</tr>
<tr>
<td>35 - 40</td>
<td>29161.</td>
<td>0.011590</td>
<td>0.45530</td>
<td>0.940870</td>
<td>1.74580</td>
<td>1.32128</td>
</tr>
<tr>
<td>40 - 45</td>
<td>42942.</td>
<td>0.018400</td>
<td>0.77390</td>
<td>0.929970</td>
<td>2.10864</td>
<td>1.45211</td>
</tr>
<tr>
<td>45 - 50</td>
<td>64283.</td>
<td>0.029020</td>
<td>1.27206</td>
<td>0.912860</td>
<td>2.70107</td>
<td>1.64349</td>
</tr>
<tr>
<td>50 - 55</td>
<td>90593.</td>
<td>0.045710</td>
<td>2.20093</td>
<td>0.886370</td>
<td>3.60661</td>
<td>1.89910</td>
</tr>
<tr>
<td>55 - 60</td>
<td>116753.</td>
<td>0.065770</td>
<td>3.46131</td>
<td>0.845850</td>
<td>5.01355</td>
<td>2.23909</td>
</tr>
<tr>
<td>60 - 65</td>
<td>153444.</td>
<td>0.102570</td>
<td>6.15306</td>
<td>0.790220</td>
<td>6.85223</td>
<td>2.61767</td>
</tr>
<tr>
<td>65 - 70</td>
<td>196605.</td>
<td>0.147630</td>
<td>9.44893</td>
<td>0.709170</td>
<td>9.36099</td>
<td>3.05957</td>
</tr>
<tr>
<td>70 - 75</td>
<td>223707.</td>
<td>0.214720</td>
<td>16.18415</td>
<td>0.604480</td>
<td>11.55334</td>
<td>3.39902</td>
</tr>
<tr>
<td>75 - 80</td>
<td>219978.</td>
<td>0.312800</td>
<td>30.56591</td>
<td>0.474690</td>
<td>13.03838</td>
<td>3.61087</td>
</tr>
<tr>
<td>80 - 85</td>
<td>185231.</td>
<td>0.463120</td>
<td>62.16667</td>
<td>0.326210</td>
<td>13.04497</td>
<td>3.61178</td>
</tr>
<tr>
<td>85 - 90</td>
<td>120366.</td>
<td>0.614370</td>
<td>120.92803</td>
<td>0.175140</td>
<td>10.37583</td>
<td>3.22115</td>
</tr>
<tr>
<td>90 - 95</td>
<td>50278.</td>
<td>0.788120</td>
<td>261.75601</td>
<td>0.067540</td>
<td>5.25246</td>
<td>2.29182</td>
</tr>
<tr>
<td>95+</td>
<td>13882.</td>
<td>1.000000</td>
<td>0.00000</td>
<td>0.014310</td>
<td>1.42976</td>
<td>1.19572</td>
</tr>
</tbody>
</table>
Table 4

Computation of standard error of survival probability.
Total California population, 1970.

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Number of Deaths in Interval ((x_{i}, x_{i+1}))</th>
<th>Probability of Dying in Interval ((x_{i}, x_{i+1}))</th>
<th>Sample Variance of (q_{i}^{2})</th>
<th>Probability of Surviving Interval ((0, x_{i}))</th>
<th>Sample Variance of (p_{0i}^{2})</th>
<th>Standard Error of (\hat{p}_{0i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x_{1}, x_{i+1}))</td>
<td>(D_{i})</td>
<td>(q_{i})</td>
<td>(10^{8} \cdot q_{i}^{2})</td>
<td>(p_{0i})</td>
<td>(10^{8} \cdot p_{0i}^{2})</td>
<td>(10^{4} \cdot p_{0i}^{2})</td>
</tr>
<tr>
<td>0 - 1</td>
<td>6234.</td>
<td>0.018010</td>
<td>5.10937</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>1 - 5</td>
<td>1049.</td>
<td>0.003220</td>
<td>0.98522</td>
<td>0.981990</td>
<td>5.10937</td>
<td>2.26039</td>
</tr>
<tr>
<td>5 - 10</td>
<td>723.</td>
<td>0.001880</td>
<td>0.48793</td>
<td>0.978830</td>
<td>6.02660</td>
<td>2.45491</td>
</tr>
<tr>
<td>10 - 15</td>
<td>735.</td>
<td>0.001870</td>
<td>0.47487</td>
<td>0.976990</td>
<td>6.47146</td>
<td>2.54390</td>
</tr>
<tr>
<td>15 - 20</td>
<td>2054.</td>
<td>0.005640</td>
<td>1.53993</td>
<td>0.975160</td>
<td>6.90051</td>
<td>2.62688</td>
</tr>
<tr>
<td>20 - 25</td>
<td>2702.</td>
<td>0.007730</td>
<td>2.19433</td>
<td>0.969660</td>
<td>8.28727</td>
<td>2.87876</td>
</tr>
<tr>
<td>25 - 30</td>
<td>2071.</td>
<td>0.007080</td>
<td>2.40325</td>
<td>0.962160</td>
<td>10.22275</td>
<td>3.19730</td>
</tr>
<tr>
<td>30 - 35</td>
<td>1964.</td>
<td>0.008020</td>
<td>3.24870</td>
<td>0.955350</td>
<td>12.30338</td>
<td>3.50761</td>
</tr>
<tr>
<td>35 - 40</td>
<td>2588.</td>
<td>0.011190</td>
<td>4.78419</td>
<td>0.947690</td>
<td>15.07196</td>
<td>3.88226</td>
</tr>
<tr>
<td>40 - 45</td>
<td>4114.</td>
<td>0.016890</td>
<td>6.81706</td>
<td>0.937090</td>
<td>19.03349</td>
<td>4.36273</td>
</tr>
<tr>
<td>45 - 50</td>
<td>6722.</td>
<td>0.026640</td>
<td>10.27645</td>
<td>0.921260</td>
<td>24.38215</td>
<td>4.93782</td>
</tr>
<tr>
<td>50 - 55</td>
<td>8948.</td>
<td>0.040490</td>
<td>17.58000</td>
<td>0.896720</td>
<td>31.82238</td>
<td>5.64113</td>
</tr>
<tr>
<td>55 - 60</td>
<td>11942.</td>
<td>0.062070</td>
<td>30.25915</td>
<td>0.880410</td>
<td>43.43359</td>
<td>6.59041</td>
</tr>
<tr>
<td>60 - 65</td>
<td>14309.</td>
<td>0.088860</td>
<td>50.27921</td>
<td>0.807000</td>
<td>60.60944</td>
<td>7.78520</td>
</tr>
<tr>
<td>65 - 70</td>
<td>17088.</td>
<td>0.128930</td>
<td>84.73635</td>
<td>0.735290</td>
<td>82.06080</td>
<td>9.11377</td>
</tr>
<tr>
<td>70 - 75</td>
<td>19149.</td>
<td>0.180520</td>
<td>139.45783</td>
<td>0.640490</td>
<td>108.83660</td>
<td>10.43247</td>
</tr>
<tr>
<td>75 - 80</td>
<td>21325.</td>
<td>0.270390</td>
<td>250.13991</td>
<td>0.524870</td>
<td>130.29900</td>
<td>11.41485</td>
</tr>
<tr>
<td>80 - 85</td>
<td>20129.</td>
<td>0.385210</td>
<td>453.21022</td>
<td>0.382950</td>
<td>138.27254</td>
<td>11.75893</td>
</tr>
<tr>
<td>85+</td>
<td>22843.</td>
<td>1.000000</td>
<td>0.000000</td>
<td>0.235430</td>
<td>118.72215</td>
<td>10.89596</td>
</tr>
</tbody>
</table>
(5) Take the square root of the variance to obtain the standard
error of \( \hat{p}_{0i} \) and record it in Column 7.

Statistical inference about the unknown survival probability
\( (p_{0i}) \) can now be made using the standard errors in Table 3 and Table 4.
For example, the estimate of the probability of surviving from birth to age 20
is \( \hat{p}_{0,20} = .96060 \) for the total United States population, 1960, and \( \hat{p}_{0,20} = .96811 \) for the total California population, 1970. To test for the signifi-
cance of difference between these two probabilities, we compute the critical ratio

\[
Z = \frac{\hat{p}_{0,20}^{(U.S.)} - \hat{p}_{0,20}^{(Cal.)}}{\text{S.E.}(\text{diff.})}.
\]

The standard error of the difference is given by

\[
\text{S.E.}(\text{diff.}) = \sqrt{(.96799 \times 10^{-6}) + (8.28727 \times 10^{-6})}
\]

\[= 3.0422 \times 10^{-4}.\]

Substituting the numerical values of \( \hat{p}_{0,20} \) and (3.8) in (3.7),

\[
Z = \frac{.96060 - .96966}{3.0422 \times 10^{-4}} = \frac{-0.00906}{3.0422 \times 10^{-4}}
\]

\[= -29.78.\]

Based on the above findings, we conclude that a newborn who is subject
to California 1970 mortality experience has a greater probability of
surviving to age 20 than one who is subject to United States 1960 experience.

The converse is true, however, for the probability of surviving from
age 20 to age 40. Table 5 shows that \( p_{20,40}^{(U.S.)} > p_{20,40}^{(Cal.)} \) and
Remark 2. The formula for the estimate \( \hat{p}_{ij} \) in (3.5) applies to both the current life table and the cohort life table. However, the formula for the variance of \( \hat{p}_{ij} \) assumes different forms in the two cases.

In a cohort life table, \( \ell \) is the number of survivors at \( x \) of \( \ell \) individuals living at \( x \), with \( \hat{p}_{ij} \) being the proportion of surviving the interval \((x, x_i)\). Therefore, \( \hat{p}_{ij} = \ell_j/\ell \) is a binomial proportion with a sample variance given by

\[
S^2_{p_{ij}} = \frac{1}{\ell} \hat{p}_{ij}(1-\hat{p}_{ij}).
\] (3.10)

Formula (3.10) for the sample variance of \( \hat{p}_{ij} \) is equal to (3.7) in the cohort life table, where the sample variance of the proportion for each age interval \( \hat{p}_h \) is computed from

\[
S^2_{p_h} = \frac{1}{\ell_h} \hat{p}_h(1-\hat{p}_h) = \frac{\ell_{h+1}}{\ell_h} \left( 1 - \frac{\ell_{h+1}}{\ell_h} \right)
\] (3.11)

for a cohort life table, then formula (3.7) will be reduced to formula (3.10). Substituting \( \hat{p}_{ij} = \ell_j/\ell \), \( \hat{p}_h = \ell_{h+1}/\ell_h \) and (3.11) in (3.6), we have

\[
S^2_{p_{ij}} = \left[ \frac{\ell_j}{\ell_i} \right]^2 \sum_{h=i}^{j-1} \left( \frac{\ell_h}{\ell_{h+1}} \right)^2 \frac{1}{\ell_h} \frac{\ell_{h+1}}{\ell_h} \left( 1 - \frac{\ell_{h+1}}{\ell_h} \right)
\]

\[
= \left[ \frac{\ell_j}{\ell_i} \right]^2 \sum_{h=i}^{j-1} \frac{1}{\ell_{h+1}} \left( 1 - \frac{\ell_{h+1}}{\ell_h} \right)
\]

\[
= \frac{1}{\ell_i} \frac{\ell_j}{\ell_i} \left( 1 - \frac{\ell_j}{\ell_i} \right) = \frac{1}{\ell_i} \hat{p}_{ij}(1-\hat{p}_{ij}),
\] (3.12)

is required to be shown.
### Table 5

Statistical test for the significance of difference between survival probabilities of United States population, 1960, and California population, 1970.

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>United States 1960</th>
<th>California 1970</th>
<th>Difference*</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, x)</td>
<td>( \hat{p}_{ij} )</td>
<td>( 10^4 S_{\hat{p}_{ij}} )</td>
<td>( \hat{p}_{ij} )</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>(0, 20)</td>
<td>.96060</td>
<td>.98386</td>
<td>.96966</td>
</tr>
<tr>
<td>(20, 40)</td>
<td>.96811</td>
<td>1.14104</td>
<td>.96641</td>
</tr>
</tbody>
</table>

*Formula for the standard error of the difference, \( \hat{p}_{ij} \) (U.S.) - \( \hat{p}_{ij} \) (Cal.):

\[
S.E.(\text{diff.}) = \sqrt{\frac{S^2_{\hat{p}_{ij}} \text{(U.S.)}}{\hat{p}_{ij}(\text{U.S.})} + \frac{S^2_{\hat{p}_{ij}} \text{(Cal.)}}{\hat{p}_{ij}(\text{Cal.})}}
\]
4. Expectation of Life at Age $x_\alpha$, $e_\alpha$

Expectation of life at a given age is the mean future lifetime beyond this age. In the life table, there are $l_\alpha$ individuals living at age $x_\alpha$. Let the lifetime beyond $x_\alpha$ of these individuals be denoted by $Y_{\alpha k}$, for $k=1, \ldots, l_\alpha$. Their mean value

$$\overline{Y}_\alpha = \frac{1}{l_\alpha} \sum_{k=1}^{l_\alpha} Y_{\alpha k} \quad (4.1)$$

is approximately normally distributed, with an expected value of $e_\alpha$. This sample mean $\overline{Y}_\alpha$ is equal to the observed expectation of life $\hat{e}_\alpha$, or

$$\overline{Y}_\alpha = \hat{e}_\alpha \quad (4.2)$$

We now show that equation (4.2) is indeed true.

As any continuous random variable, lifetime of an individual is not accurately measured. In fact, the values of the $l_\alpha$ values are not individually recorded in the life table, but grouped in the form of a frequency distribution in which the ages $x_i$ and $x_{i+1}$ are the lower and upper limits for the interval $i$, and the deaths, $d_i$, are the corresponding frequencies for $i = \alpha, \alpha+1, \ldots, w$. The sum of the frequencies equals the number of survivors at age $x_\alpha$, or

$$d_\alpha + \ldots + d_w = l_\alpha \quad (4.3)$$

The total number of years remaining to the $l_\alpha$ survivors depends on the exact age at which death occurs, that is, on the distribution of deaths within each age interval. Suppose that the distribution of death in the interval $(x_i, x_{i+1})$ is such that, on the average, each of the $d_i$ individuals lives a fraction $a_i$ of the interval, or $a_i n_i$ years in the interval (since $x_{i+1} - x_i = n_i$ is the interval length). Each thus lives
where \( x_i + \alpha \) years, or \( x_i - x + \alpha \) years after age \( x \), and the sample mean
is given by

\[
\bar{y}_\alpha = \frac{1}{x} \sum_{i=\alpha}^w (x_i - x_\alpha + \alpha n_i) \ d_i
\]

By definition

\[
x_i - x_\alpha = n_\alpha n_{\alpha+1} + \ldots + n_{i-1} = \sum_{j=\alpha}^{i-1} n_j
\]

hence

\[
\sum_{i=\alpha}^w (x_i - x_\alpha) \ d_i = \sum_{i=\alpha}^w \left( \sum_{j=\alpha}^{i-1} n_j \right) \ d_i
\]

\[
= \sum_{j=\alpha}^{i-1} n_j \ \sum_{i=j+1}^w d_i
\]

Since the number of individuals living at age \( x_j \) will all eventually die,

\[
\bar{d}_j = d_j + d_{j+1} + \ldots + d_w
\]

or

\[
\bar{d}_j - d_j = d_{j+1} + \ldots + d_w = \sum_{i=j+1}^w d_i
\]

Therefore (4.6) may be rewritten

\[
\sum_{i=\alpha}^w (x_i - x_\alpha) \ d_i = \sum_{j=\alpha}^{i-1} n_j (\bar{d}_j - d_j)
\]

Substituting (4.9) in (4.4) gives
The quantity inside the braces, for $i = \alpha, \ldots, w-1$,
\[ n_i (\ell_i - d_i) + a_i n_i d_i = L_i , \quad (4.11) \]
is the number of years lived by the $n_i$ individuals in the interval $(x_i, x_{i+1})$.

Also we let
\[ L_w = a_{n_w} d_w = a_{n_w} L_w \quad (4.12) \]
be the number of years lived by $L_{w}$ beyond age $x_w$.

Using (4.11) and (4.12) we rewrite (4.10) as
\[ \bar{\gamma}_\alpha = \frac{L_\alpha + L_{\alpha+1} + \ldots + L_w}{L_\alpha} \quad (4.13) \]
which, of course, is $e_\alpha$, the observed expectation of life at age $x_\alpha$.

4.1 Formula for the variance of the expectation of life. Once the equality between the observed expectation of life $e_\alpha$ and the sample mean future lifetime $\bar{\gamma}_\alpha$ is established, the sample variance of $e_\alpha$ can easily be computed. We may visualize the age and death columns in a cohort life table as a frequency distribution, with $x_i - x + a_i n_i$ being the average value and $d_i$ the corresponding frequency so that the sample variance of $Y_\alpha$ is given by
Consequently, we have the formula for the sample variance of $\tilde{\nu}$ (or $\hat{e}_\alpha$)

$$S_{\tilde{\nu}}^2 = S_{\tilde{\nu}}^2 = \frac{1}{n-1} \sum_{i=\alpha}^n [(x_i - x + a.n.) - \bar{e}_\alpha]^2 d_i .$$  (4.14)$$

or, by substitution of (4.14),

$$S_{\tilde{\nu}}^2 = \frac{1}{n-1} \sum_{i=\alpha}^n [(x_i - x + a.n.) - \bar{e}_\alpha]^2 d_i .$$  (4.15)$$

In formula (4.14), $a_i, n_i, \bar{e}_\alpha, d_i$ and $l_\alpha$ are all given in a life table; the sample variance of $\hat{e}_\alpha$ can be determined.

Formula (4.14), however, is not applicable for the current life table for a number of reasons. First of all, figures $d_i$ and $l_\alpha$ are dependent upon the choice of the radix $l_0$, and therefore are not meaningful quantities when they appear without reference to $l_0$. Secondly, basic variables in a current life table are the $\hat{q}_i$. Therefore, the sample variance estimates of $\hat{e}_\alpha$ should be expressed in terms of the variance of $\hat{q}_i$.

Formula (4.10) for the observed expectation of life, with the substitution of $l_{j+1} = l_j - d_j$ and $l_j - l_j+1 = d_j$, may be rewritten as

$$\hat{e}_\alpha = \frac{1}{l_\alpha} \left[ \sum_{j=\alpha}^{w-1} \left( n_j \hat{l}_{j+1} + a_j n_j \hat{l}_j (\hat{l}_j - \hat{l}_j+1) \right) + a_n d \right] .$$  (4.15)$$

or

$$\hat{e}_\alpha = \frac{1}{l_\alpha} \left[ \sum_{j=\alpha+1}^{w} \left( (1-a_j-1)n_{j-1} + a_j n_j \right) \hat{l}_j \right] .$$  (4.16)$$

Now we let

$$c_j = (1-a_j-1)n_{j-1} + a_j n_j$$  (4.17)$$
and write
\[
\hat{e}_a = a_n a + \frac{c_{a+1} \cdot \xi_{a+1} + \cdots + c_w \cdot \xi_w}{\xi_a}
\]  \hspace{1cm} (4.18)

or, since \( \xi_j / \xi_a = \hat{p}_{aj} \),
\[
\hat{e}_a = a_n a + \sum_{j=a+1}^{w} c_j \hat{p}_{aj} \]  \hspace{1cm} (4.19)

Thus, the observed expectation of life \( \hat{e}_a \) is a linear function of \( \hat{p}_{aj} \),
which in the current life table is computed from
\[
\hat{p}_{aj} = \hat{p}_a \hat{p}_{a+1} \cdots \hat{p}_{j-1}, \quad j = a+1, \ldots, w .
\]  \hspace{1cm} (4.20)

Clearly, the derivatives of \( \hat{p}_{aj} \) with respect to \( \hat{p}_i \) is given by
\[
\frac{\partial}{\partial \hat{p}_i} \hat{p}_{aj} = \hat{p}_{ai} \hat{p}_{i+1,j} \quad \text{for } a < i < j
\]
\[= 0 \quad \text{otherwise} \]  \hspace{1cm} (4.21)

Hence, from (4.19)
\[
\frac{\partial}{\partial \hat{p}_i} \hat{e}_a = \sum_{j=a+1}^{w} c_j \frac{\partial}{\partial \hat{p}_i} \hat{p}_{aj}
\]
\[= \sum_{j=i+1}^{w} c_j \hat{p}_{ai} \hat{p}_{i+1,j} \]
\[= p_{ai} \left( c_{i+1} + \sum_{j=i+2}^{w} c_j \hat{p}_{i+1,j} \right) \]  \hspace{1cm} (4.22)

Using relation (4.19), or
\[ \hat{e}_{i+1} = a_{i+1}n_{i+1} + \sum_{j=i+2}^{w} c_{j} p_{i+1,j}, \]  
\[ c_{i+1} = (1-a_{i})n_{i} + a_{i+1}n_{i+1} \]  

we rewrite the derivative in (4.22) as follows
\[ \frac{\partial}{\partial \hat{p}_{i}} \hat{e}_{\alpha} = \hat{p}_{\alpha i} [(1-a_{i})n_{i} + \hat{e}_{i+1}] . \]  

Because of (4.21), the derivative (4.25) vanishes when \( i = w \). Now the estimated probabilities (\( \hat{p}_{i} \)) for two nonoverlapping age intervals are based on mortality experience of two distinct groups of people, and therefore are not correlated. Consequently, the variance of the expectation of life may be computed from the following
\[ S_{\hat{e}_{\alpha}}^{2} = \sum_{i=\alpha}^{w-1} \left( \frac{\partial}{\partial \hat{p}_{i}} \hat{e}_{\alpha} \right)^{2} S_{\hat{p}_{i}}^{2} . \]  

Substituting (4.25) in (4.26) yields the desired formula for the sample variance of \( \hat{e}_{\alpha} \):
\[ S_{\hat{e}_{\alpha}}^{2} = \sum_{i=\alpha}^{w-1} \hat{p}_{\alpha i}^{2} [(1-a_{i})n_{i} + \hat{e}_{i+1}]^{2} S_{\hat{p}_{i}}^{2} . \]  

where the variance of \( \hat{p}_{i} \) is given in (2.2).

\[ \frac{S_{\hat{p}_{i}}^{2}}{\hat{p}_{i}} = \frac{\hat{q}_{i}^{2}(1-\hat{q}_{i})}{D_{i}} . \]  

4.2 Computation of the variance of the expectation of life in a current life table. Formula (4.27), which holds true for any age \( x_{\alpha} \) in the life table, contains terms that appear repeatedly for different values of \( \alpha \). Therefore, the variances of \( \hat{e}_{\alpha} \) for all ages \( x_{\alpha} \) in the life table can be
calculated by a single computation program. Using formula (4.27) and referring to Table 6, the essential steps in the computation of the sample variance of $\hat{e}_\alpha$ are as follows.

1. Designate age interval in Column 1.

2. Record the length of age interval $n_i$ in Column 2, and the fraction of last age interval of life $a_i$ in Column 3.

3. Compute the sample variance of $\hat{p}_i$ ($\hat{q}_i$) from formula (2.2) and record it in Column 4.

4. Compute for each age interval the quantity

$$\left(\frac{1}{\hat{q}_i^2} + \frac{\hat{p}_i}{\hat{q}_i}\right)^2$$

and record it in Column 5.

5. Sum the products in Column 5 from the bottom of the table up to $x_\alpha$ and enter the sum in Column 6.

6. Divide the sum in Column 6 by $\hat{q}_i^2$ to obtain the sample variance of the observed expectation of life in Column 7.

7. Take the square root of the sample variance to obtain the sample standard error of the observed expectation of life, as shown in Column 8.

4.3 Statistical inference about expectation of life. An observed expectation of life, as shown earlier in this section, is a sample mean of future lifetime. Therefore, statistical tests based on normal distribution may be used in making inference regarding expectation of life at a particular age, or in comparing expectation of life of two or more populations. In Table 7 the expectations of life for the United States 1960 population are compared with those for the California 1970 population. For each age the expectation of life and the standard errors are recorded in Columns 2 through 5. The difference of the expectations is given in Column 6.
The standard error of the difference computed from

\[ \text{S.E.}(\text{diff.}) = \sqrt{S^2_{\text{Cal.}} + S^2_{\text{U.S.}}} \]  

is recorded in Column (7). The ratio of the difference to the corresponding standard error is recorded in Column 8.

The critical ratio for each age far exceeds the critical value of 2.33 in the normal distribution corresponding to the one percent level of significance. This means that a person of any age, who is subject to the California 1970 mortality experience, has a greater expectation of life than one who is subject to the United States 1960 experience.
Table 6

Computation of the sample variance of the observed expectation of life

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>Length of interval</th>
<th>Fraction of last age interval of life</th>
<th>Sample variance of $\hat{\beta}_1$</th>
<th>$s^2_{\beta_1}$</th>
<th>Sample variance of $\hat{\alpha}_1$</th>
<th>$s^2_{\alpha_1}$</th>
<th>Sample Standard Error of $\hat{\alpha}_1$</th>
<th>$s_{\alpha_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>1</td>
<td>.10</td>
<td>.6043</td>
<td>308,328.66</td>
<td>1,384,473.52</td>
<td>1.3845</td>
<td>.012</td>
<td></td>
</tr>
<tr>
<td>1-5</td>
<td>4</td>
<td>.39</td>
<td>.1070</td>
<td>48,684.08</td>
<td>1,076,144.86</td>
<td>1.1349</td>
<td>.011</td>
<td></td>
</tr>
<tr>
<td>5-10</td>
<td>5</td>
<td>.46</td>
<td>.0653</td>
<td>25,686.27</td>
<td>1,027,460.78</td>
<td>1.0931</td>
<td>.010</td>
<td></td>
</tr>
<tr>
<td>10-15</td>
<td>5</td>
<td>.54</td>
<td>.0649</td>
<td>21,433.28</td>
<td>1,001,774.51</td>
<td>1.0710</td>
<td>.010</td>
<td></td>
</tr>
<tr>
<td>15-20</td>
<td>5</td>
<td>.57</td>
<td>.1714</td>
<td>47,445.51</td>
<td>980,341.23</td>
<td>1.0527</td>
<td>.010</td>
<td></td>
</tr>
<tr>
<td>20-25</td>
<td>5</td>
<td>.49</td>
<td>.2825</td>
<td>65,770.32</td>
<td>932,895.72</td>
<td>1.0110</td>
<td>.010</td>
<td></td>
</tr>
<tr>
<td>25-30</td>
<td>5</td>
<td>.50</td>
<td>.2971</td>
<td>55,984.55</td>
<td>867,125.40</td>
<td>.9514</td>
<td>.010</td>
<td></td>
</tr>
<tr>
<td>30-35</td>
<td>5</td>
<td>.52</td>
<td>.3307</td>
<td>49,278.91</td>
<td>811,140.85</td>
<td>.9017</td>
<td>.009</td>
<td></td>
</tr>
<tr>
<td>35-40</td>
<td>5</td>
<td>.54</td>
<td>.4553</td>
<td>52,293.09</td>
<td>761,861.94</td>
<td>.8606</td>
<td>.009</td>
<td></td>
</tr>
<tr>
<td>40-45</td>
<td>5</td>
<td>.54</td>
<td>.7739</td>
<td>66,833.91</td>
<td>709,568.85</td>
<td>.8205</td>
<td>.009</td>
<td></td>
</tr>
<tr>
<td>45-50</td>
<td>5</td>
<td>.54</td>
<td>1.2721</td>
<td>79,526.83</td>
<td>642,734.94</td>
<td>.7713</td>
<td>.009</td>
<td></td>
</tr>
<tr>
<td>50-55</td>
<td>5</td>
<td>.53</td>
<td>2.2009</td>
<td>95,654.42</td>
<td>563,208.11</td>
<td>.7169</td>
<td>.008</td>
<td></td>
</tr>
<tr>
<td>55-60</td>
<td>5</td>
<td>.52</td>
<td>3.4613</td>
<td>97,872.06</td>
<td>467,553.69</td>
<td>.6535</td>
<td>.008</td>
<td></td>
</tr>
<tr>
<td>60-65</td>
<td>5</td>
<td>.52</td>
<td>6.1531</td>
<td>105,623.14</td>
<td>369,681.63</td>
<td>.5920</td>
<td>.008</td>
<td></td>
</tr>
<tr>
<td>65-70</td>
<td>5</td>
<td>.52</td>
<td>9.4489</td>
<td>87,635.79</td>
<td>264,058.49</td>
<td>.5250</td>
<td>.007</td>
<td></td>
</tr>
<tr>
<td>70-75</td>
<td>5</td>
<td>.51</td>
<td>16.1842</td>
<td>71,425.04</td>
<td>176,422.70</td>
<td>.4828</td>
<td>.007</td>
<td></td>
</tr>
<tr>
<td>75-80</td>
<td>5</td>
<td>.51</td>
<td>30.5659</td>
<td>52,370.93</td>
<td>104,997.66</td>
<td>.4660</td>
<td>.007</td>
<td></td>
</tr>
<tr>
<td>80-85</td>
<td>5</td>
<td>.48</td>
<td>62.1657</td>
<td>34,293.39</td>
<td>52,626.73</td>
<td>.4946</td>
<td>.007</td>
<td></td>
</tr>
<tr>
<td>85-90</td>
<td>5</td>
<td>.45</td>
<td>120.9280</td>
<td>13,802.48</td>
<td>18,333.34</td>
<td>.5977</td>
<td>.008</td>
<td></td>
</tr>
<tr>
<td>90-95</td>
<td>5</td>
<td>.41</td>
<td>261.7560</td>
<td>4,530.86</td>
<td>4,530.86</td>
<td>.9932</td>
<td>.010</td>
<td></td>
</tr>
</tbody>
</table>
Table 7


<table>
<thead>
<tr>
<th>Age Interval</th>
<th>United States</th>
<th>California</th>
<th>Difference</th>
<th>Critical Ratio *</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{e}_i )</td>
<td>( S_{\hat{e}_i} )</td>
<td>( \hat{e}_i )</td>
<td>( S_{\hat{e}_i} )</td>
</tr>
<tr>
<td></td>
<td>( (1) )</td>
<td>( (2) )</td>
<td>( (3) )</td>
<td>( (4) )</td>
</tr>
<tr>
<td>0-1</td>
<td>69.65</td>
<td>0.012</td>
<td>71.95</td>
<td>0.037</td>
</tr>
<tr>
<td>1-5</td>
<td>70.53</td>
<td>0.011</td>
<td>72.27</td>
<td>0.034</td>
</tr>
<tr>
<td>5-10</td>
<td>66.83</td>
<td>0.010</td>
<td>68.50</td>
<td>0.033</td>
</tr>
<tr>
<td>10-15</td>
<td>61.99</td>
<td>0.010</td>
<td>63.62</td>
<td>0.033</td>
</tr>
<tr>
<td>15-20</td>
<td>57.12</td>
<td>0.010</td>
<td>58.74</td>
<td>0.033</td>
</tr>
<tr>
<td>20-25</td>
<td>52.37</td>
<td>0.010</td>
<td>54.05</td>
<td>0.032</td>
</tr>
<tr>
<td>25-30</td>
<td>47.68</td>
<td>0.010</td>
<td>49.46</td>
<td>0.032</td>
</tr>
<tr>
<td>30-35</td>
<td>42.97</td>
<td>0.009</td>
<td>44.79</td>
<td>0.031</td>
</tr>
<tr>
<td>35-40</td>
<td>38.30</td>
<td>0.009</td>
<td>40.13</td>
<td>0.030</td>
</tr>
<tr>
<td>40-45</td>
<td>33.72</td>
<td>0.009</td>
<td>35.56</td>
<td>0.030</td>
</tr>
<tr>
<td>45-50</td>
<td>29.30</td>
<td>0.009</td>
<td>31.12</td>
<td>0.030</td>
</tr>
<tr>
<td>50-55</td>
<td>25.09</td>
<td>0.008</td>
<td>26.90</td>
<td>0.029</td>
</tr>
<tr>
<td>55-60</td>
<td>21.17</td>
<td>0.008</td>
<td>22.92</td>
<td>0.028</td>
</tr>
<tr>
<td>60-65</td>
<td>17.48</td>
<td>0.008</td>
<td>19.27</td>
<td>0.027</td>
</tr>
<tr>
<td>65-70</td>
<td>14.18</td>
<td>0.007</td>
<td>15.89</td>
<td>0.026</td>
</tr>
<tr>
<td>70-75</td>
<td>11.18</td>
<td>0.007</td>
<td>12.87</td>
<td>0.024</td>
</tr>
<tr>
<td>75-80</td>
<td>8.54</td>
<td>0.007</td>
<td>10.13</td>
<td>0.023</td>
</tr>
<tr>
<td>80-85</td>
<td>6.27</td>
<td>0.007</td>
<td>7.94</td>
<td>0.021</td>
</tr>
</tbody>
</table>

*Formula for the standard error of the difference \( \hat{e}_i \text{(Cal.)} - \hat{e}_i \text{(U.S.)} \)

\[
S.E.(\text{diff.}) = \sqrt{\frac{S_{\hat{e}_i \text{(Cal.)}}^2}{\hat{e}_i \text{(Cal.)}} + \frac{S_{\hat{e}_i \text{(U.S.)}}^2}{\hat{e}_i \text{(U.S.)}}}
\]
1. Introduction

The multiple decrement table is not only a useful means of summarizing mortality experience of a defined population subject to several risks of dying, but also a powerful analytical tool for the study of decrement data. The concept of multiple decrement originates in the investigation of component causes of death; however, it has applications in many research fields. In the actuarial sciences, for example, disability and mortality are distinct causes of claim, and the effects of exposure to both causes and their interaction must be analyzed in a meaningful way. Dissolution of a marriage may be because of death occurring to either one of the partners or because of divorce. Here there are three forces of decrement; death to the male, death to the female and divorce. Similarly, survival of an enterprise is subject to many forces of decrement and their interacting effects. In spite of numerous applications of this methodology, the most important use of multiple decrement tables still remains to be in the study of mortality.

The multiple decrement table is directly related to the theory of competing risks presented in Appendix III. The theory has been developed to evaluate the forces of mortality of competing risks under investigation. According to the theory, there are three types of probability of death with respect to a particular risk or risks.

1.1. Crude probability: The probability of death from a specific cause in the presence of competition of all other risks acting in a population.
1.2. Net probability: The probability of death if a specific risk is the only risk in effect in the population or, conversely, the probability of death if a specific risk is eliminated from the population.

1.3. Partial crude probability: The probability of death from a specific cause when another risk (or risks) is eliminated from the population.

Detailed discussion of these probabilities is presented in Appendix III. Clarification should be made of the terms "risk" and "cause." Both terms may refer to the same condition but are different on the time scale relative to the occurrence of death. Prior to death the condition in question is a risk; after death the condition is a cause (provided, of course, this is the condition from which an individual dies). We shall take up this point again in Appendix III.

An ordinary multiple decrement table contains only the crude probability of death for selected causes covering the entire life span of a well defined population. For easy comparison, the probability of death \( q_1 \) without referring to causes is often included. Let \( q_{i\delta} \) be the probability that an individual alive at exact age \( x_i \) will die in interval \( (x_i, x_{i+1}) \) from cause \( R_\delta \) in the presence of all other competing risks, for \( i = 0, 1, \ldots, w; \) \( \delta = 1, \ldots, r \). A typical decrement table is given in Table 1 on page 5-3.

There are two types of multiple decrement tables for the analysis of human mortality: The cohort multiple decrement table and the current multiple decrement table. As in the case of the life table, a cohort multiple decrement table records the mortality experience of a well-defined cohort of people from the birth of each person to the death of the last person of the group. When a cohort of people is subject to a number of risks of dying, there will be deaths from each of these risks within every age interval of life. The number of deaths from a specific cause (say \( R_\delta \)) in an interval \( (x_i, x_{i+1}) \) divided by the number of individuals alive at the
<table>
<thead>
<tr>
<th>Age Interval ( (x_i, x_{i+1}) )</th>
<th>Probability of Dying in Interval ( (x_i, x_{i+1}) )</th>
<th>Causes of Death</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>( q_0 )</td>
<td>( q_{01} ) ( \ldots ) ( q_{0r} )</td>
</tr>
<tr>
<td>1-5</td>
<td>( q_1 )</td>
<td>( q_{11} ) ( \ldots ) ( q_{1r} )</td>
</tr>
<tr>
<td>( x_i-x_{i+1} )</td>
<td>( q_i )</td>
<td>( q_{i1} ) ( \ldots ) ( q_{ir} )</td>
</tr>
<tr>
<td>( x_w &amp; \text{over} )</td>
<td>( q_w )</td>
<td>( q_{w1} ) ( \ldots ) ( q_{wr} )</td>
</tr>
</tbody>
</table>
beginning of the interval is an estimate of the (crude) probability of
dying from cause $R_6$, denoted by $\hat{Q}_{i6}$. This estimate is simply the proportion
of individuals dying from a specific cause in a defined age interval. An
aggregate of these proportions for different causes of death over all age
intervals forms a cohort multiple decrement table. A detailed discussion
and theoretical aspects of the table may be found in Appendix IV of this
manual.

A current multiple decrement table, which is more useful for practical
purposes, is the one derived from the mortality experience of a population
of all ages over a short period of time, such as one year. The appearance
of this table is exactly the same as the cohort multiple decrement table,
but differs from the latter in the basic information from which the table
is constructed. Specifically, the data for the current decrement table are
the number of deaths from different causes and the corresponding mid-year
population for each age group over the entire life span of a current population,
from which age-and-cause specific death rates are computed. These rates in
turn are then used to compute the estimate of the (crude) probability of
dying ($\hat{Q}_{i6}$) from each cause $R_6$. A brief description is presented below.

2. Computation of the Crude Probability, $\hat{Q}_{i6}$

Let us first reintroduce the symbols used in Chapter 3. For age interval
$(x_i, x_{i+1})$ we let $n_i = x_{i+1} - x_i$ be the length of the interval, $P_i$ the midyear
population, $D_i$ the number of deaths occurring during the calendar year, $a_i$
the fraction of the last age interval lived by each of the $D_i$ individuals,
and $N_i$ the number of people alive at $x_i$ among whom $D_i$ deaths occur. The age-
specific death rate is defined by the ratio of $D_i$ to the number of years lived
by the $N_i$ people in the interval $(x_i, x_{i+1})$, or

$$M_i = \frac{D_i}{(N_i - D_i)n_i + a_i n_i D_i} \tag{2.1}$$
When the denominator is estimated with the mid-year population, \( P_i \),

\[
(N_i - D_i) n_{i} + a_i n_{i} D_i = P_i
\]  
(2.2)

we have

\[
M_i = \frac{D_i}{P_i}.
\]  
(2.3)

The probability of dying in the interval \((x_i, x_{i+1})\) is estimated by

\[
\hat{q}_i = \frac{D_i}{N_i},
\]  
(2.4)

where \( N_i \) can be derived from (2.2), or

\[
N_i = \frac{1}{n_i} P_i [1 + (1 - a_i) n_i M_i].
\]  
(2.2a)

Substituting (2.2a) in (2.4) gives

\[
\hat{q}_i = \frac{n_i M_i}{1 + (1 - a_i) n_i M_i}.
\]  
(2.5)

The \( D_i \) deaths are now further divided according to cause with \( D_{i\delta} \)
dying from cause \( R_\delta \), \( \delta = 1, \ldots, r \), and

\[
D_i = D_{i1} + \ldots + D_{ir}
\]  
(2.6)

so that

\[
M_{i\delta} = \frac{D_{i\delta}}{P_i}, \quad \delta = 1, \ldots, r,
\]  
(2.7)

are age-cause-specific death rates. The estimate of the crude probability

of dying from \( R_\delta \) in the presence of competing risks is obviously

\[
\hat{Q}_{i\delta} = \frac{D_{i\delta}}{N_i}.
\]  
(2.8)

Substituting (2.2a) in (2.8) gives the formula for the crude probability

\[
\hat{Q}_{i\delta} = \frac{n_i M_{i\delta}}{1 + (1 - a_i) n_i M_i}.
\]  
(2.9)
We see from (2.4) and (2.8) that \( q_{i\delta} \) can be computed also from

\[
\hat{q}_{i\delta} = \frac{D_{i\delta}}{D_i} \hat{q}_i. \tag{2.9a}
\]

It is easy to show that \( \hat{q}_{i\delta} \) in (2.9a) and \( \hat{q}_i \) in (2.4) satisfy the relationship

\[
\hat{q}_{i1} + \ldots + \hat{q}_{ir} = \hat{q}_i. \tag{2.10}
\]

The formula for the sample variance of the estimator \( \hat{q}_{i\delta} \) can be derived from

\[
\text{Var}(\hat{q}_{i\delta}) = \frac{1}{N_i} \hat{q}_{i\delta}(1 - \hat{q}_{i\delta}) \tag{2.11}
\]

by substituting \( \hat{q}_{i\delta} \) for \( q_{i\delta} \) and using (2.8):

\[
\text{Var}(\hat{q}_{i\delta}) = \frac{1}{N_i} \hat{q}_{i\delta}(1 - \hat{q}_{i\delta}) = \frac{1}{N_i} \hat{q}_{i\delta}^2(1 - \hat{q}_{i\delta}). \tag{2.12}
\]

The standard deviation of \( \hat{q}_{i\delta} \) is the square root of the variance in (2.12).

The steps involved in constructing a multiple decrement table may be summarized as follows:

2.1. Information needed from a current population.

(a) Number of deaths in each age interval \((x_i, x_{i+1})\) from each cause \( D_{i\delta} \), \( D_{i1} \), and the total number of deaths \( D_i \), with

\[
D_i = D_{i1} + \ldots + D_{ir}. \tag{2.6}
\]

(b) Mid-year population for each age interval \((x_i, x_{i+1})\), \( P_i \)

(c) The fraction of the last age interval of life, \( a_x \), as given in Appendix V.
2.2. Computation of rates and probabilities

(a) Age-cause-specific death rate:

\[ M_{id} = \frac{D_{id}}{P_{i}} \] \hspace{1cm} (2.7)

for each age and each cause; and the age-specific death rate:

\[ M_i = \frac{D_i}{P_i} \] \hspace{1cm} (2.3)

(b) The probability of dying:

\[ \hat{q}_i = \frac{n_i M_i}{1 + (1-a_i)n_i M_i} \] \hspace{1cm} (2.5)

and the crude probability of dying from \( n_i \) for age interval \( (x_i, x_{i+1}) \):

\[ \hat{q}_{id} = \frac{D_{id}}{D_i} \hat{q}_i \] \hspace{1cm} (2.9a)

2.3. Computation of the standard deviation:

\[ \text{S.D.}(\hat{q}_{i\delta}) = \sqrt{\frac{1}{n_{i\delta}} \hat{q}_{i\delta}^2 (1-\hat{q}_{i\delta})} \] \hspace{1cm} (2.13)

and

\[ \text{S.D.}(q_i) = \sqrt{\frac{1}{n_i} \hat{q}_i^2 (1-\hat{q}_i)} \] \hspace{1cm} (2.14)
To illustrate the computation, let us consider as an example the competing risks of death: cardiovascular diseases ($R_1$), cancer all forms ($R_2$), all accidents ($R_3$), infectious diseases ($R_4$), respiratory diseases ($R_5$), motor vehicle accidents ($R_6$), and all other causes ($R_7$) in the Sweden population age group (1,5), 1967 in Table 2. During 1967 there were a total of $D_1 = 250$ deaths occurring in the Sweden population between age one and five; and a mid-year population of $P_1 = 471,119$. The number of deaths, $D_1 = 250$, is entered in column (2) to the right of "all causes". Dividing $D_1$ by the mid-year population $P_1 = 471,119$ gives the age specific death rate $M_1 = .000531$ entered in column (3) on the same line. The fraction of last age interval of life is $a_1 = .43$ in this case and the age interval is $n_1 = 4$ years. With these values, we compute the probability of dying for this age interval as before from

$$q^u = \frac{n_{i1} M_1}{1 + (1-a_1) n_{i1} M_1}$$

$$q^u = \frac{4(.000531)}{1 + (1-.43)4(.000531)} = .002121$$

which is recorded in column (4). The number of deaths are then identified by cause, with $D_{i1} = 4$ deaths from cardiovascular diseases, etc. These numbers are entered in column (2). Dividing each of these values by the mid-year population $P_1$, we obtain the death rate specific for the cause in question as shown in column (3). The crude probability of dying from a specific cause when all other competing causes are acting may be computed from the corresponding death rate. But it is more convenient to use the relation

$$q_{i\delta} = \frac{D_{i\delta}}{D_1} q^u_i, \text{ for } \delta = 1,2,\ldots,r \quad (2.9a)$$
Thus, for $R_2$: cancer all forms, for example,

$$
\hat{q}_{12} = \frac{D_{12}}{n_1} \hat{q} = \frac{46}{250} = .002121 = .000390
$$

These crude probabilities are shown in column (4).

The standard errors of these probabilities are computed from

$$
S.D.(\hat{q}_i) = \sqrt{\frac{1}{n_i} \hat{q}_i^2 (1-\hat{q}_i)} = .0001340 \tag{2.14}
$$

and

$$
S.D.(\hat{0}_{i\delta}) = \sqrt{\frac{1}{n_i} \hat{0}_{i\delta}^2 (1-\hat{0}_{i\delta})} \tag{2.13}
$$

for $\delta = 1, 2, \ldots, 7$. The numerical values are recorded in column (6) of Table 2.

Such computations, which can be carried out easily with computers, are needed for each age group. The basic data, the mid-year population and the number of deaths by age interval and cause, for Sweden and Australia are given in Tables 3 and 4 respectively.
Table 2

Computation of the crude probability of dying from a specific cause and the corresponding standard error. Sweden population, age interval (1, 5), 1967

<table>
<thead>
<tr>
<th>Cause of death</th>
<th>Number of deaths</th>
<th>Cause Specific Death Rate</th>
<th>Crude Probability of dying</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R_8</td>
<td>D_18</td>
<td>M_18</td>
<td>A_18</td>
<td>S.D. (A_18)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>All causes</td>
<td>250</td>
<td>0.000531</td>
<td>0.002121</td>
<td>0.0001340</td>
</tr>
<tr>
<td>R_1: Cardiovascular</td>
<td>4</td>
<td>0.000008</td>
<td>0.000034</td>
<td>0.000169</td>
</tr>
<tr>
<td>diseases</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R_2: Cancer all forms</td>
<td>46</td>
<td>0.000098</td>
<td>0.000390</td>
<td>0.000575</td>
</tr>
<tr>
<td>R_3: All accidents</td>
<td>68</td>
<td>0.000144</td>
<td>0.000577</td>
<td>0.000700</td>
</tr>
<tr>
<td>R_4: Infectious</td>
<td>14</td>
<td>0.000030</td>
<td>0.000119</td>
<td>0.000318</td>
</tr>
<tr>
<td>diseases</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R_5: Respiratory</td>
<td>37</td>
<td>0.000079</td>
<td>0.000314</td>
<td>0.000516</td>
</tr>
<tr>
<td>diseases</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R_6: Motor vehicle</td>
<td>19</td>
<td>0.000040</td>
<td>0.000161</td>
<td>0.000369</td>
</tr>
<tr>
<td>accidents</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R_7: All other causes</td>
<td>81</td>
<td>0.000172</td>
<td>0.000687</td>
<td>0.000763</td>
</tr>
</tbody>
</table>

1/ \( q_i = \frac{n_i M_i}{1 + (1-a_i) n_i M_i} \)

2/ \( \frac{\Delta_i}{\Delta_i} = \frac{D_1 D_i}{\Delta_i} \)

3/ \( \text{S.D.} = \sqrt{\frac{1 - \Delta_i^2 (1 - \frac{\Delta}{\Delta_i})}{D_1 \Delta_i}} \)

4/ \( D_1 = 250 \)

\( q_1 = 0.002121 \)
Table 3
Mid-year population and deaths by age and cause
Sweden, 1967

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
<th>Total Deaths</th>
<th>Fraction of last age interval of life</th>
<th>Cause of Death</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>CVD</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( R_1 )</td>
</tr>
<tr>
<td>(!x_i, x_{i+1}! )</td>
<td>( p_i )</td>
<td>( d_i )</td>
<td>( a_i )</td>
<td>( d_{11} )</td>
</tr>
<tr>
<td>0-1</td>
<td>120905</td>
<td>1560</td>
<td>0.08</td>
<td>7</td>
</tr>
<tr>
<td>1-5</td>
<td>471119</td>
<td>250</td>
<td>0.43</td>
<td>4</td>
</tr>
<tr>
<td>5-10</td>
<td>522261</td>
<td>171</td>
<td>0.45</td>
<td>5</td>
</tr>
<tr>
<td>10-15</td>
<td>534756</td>
<td>148</td>
<td>0.52</td>
<td>8</td>
</tr>
<tr>
<td>15-20</td>
<td>589158</td>
<td>318</td>
<td>0.56</td>
<td>22</td>
</tr>
<tr>
<td>20-25</td>
<td>656338</td>
<td>508</td>
<td>0.050</td>
<td>23</td>
</tr>
<tr>
<td>25-30</td>
<td>510785</td>
<td>476</td>
<td>0.52</td>
<td>27</td>
</tr>
<tr>
<td>30-35</td>
<td>445412</td>
<td>517</td>
<td>0.53</td>
<td>47</td>
</tr>
<tr>
<td>35-40</td>
<td>462977</td>
<td>683</td>
<td>0.53</td>
<td>95</td>
</tr>
<tr>
<td>40-45</td>
<td>506480</td>
<td>1157</td>
<td>0.53</td>
<td>228</td>
</tr>
<tr>
<td>45-50</td>
<td>543670</td>
<td>1853</td>
<td>0.54</td>
<td>482</td>
</tr>
<tr>
<td>50-55</td>
<td>516154</td>
<td>2724</td>
<td>0.54</td>
<td>912</td>
</tr>
<tr>
<td>55-60</td>
<td>511489</td>
<td>4266</td>
<td>0.53</td>
<td>1742</td>
</tr>
<tr>
<td>60-65</td>
<td>446800</td>
<td>6189</td>
<td>0.53</td>
<td>2905</td>
</tr>
<tr>
<td>65-70</td>
<td>373773</td>
<td>8770</td>
<td>0.54</td>
<td>4625</td>
</tr>
<tr>
<td>70-75</td>
<td>286391</td>
<td>11339</td>
<td>0.53</td>
<td>6501</td>
</tr>
<tr>
<td>75-80</td>
<td>19498</td>
<td>13715</td>
<td>0.52</td>
<td>8225</td>
</tr>
<tr>
<td>80-85</td>
<td>113212</td>
<td>12766</td>
<td>0.50</td>
<td>8042</td>
</tr>
<tr>
<td>85+</td>
<td>59753</td>
<td>12373</td>
<td></td>
<td>8086</td>
</tr>
</tbody>
</table>
Table 4
Mid-year population and deaths by age and cause - Australia, 1967

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
<th>Total Deaths</th>
<th>Fraction of last age interval of life</th>
<th>Cause of Death</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>D_1</td>
<td>a_1</td>
<td>R_1</td>
</tr>
<tr>
<td>(x_1,x_2)</td>
<td>p_1</td>
<td>D_11</td>
<td>D_16</td>
<td>18</td>
</tr>
<tr>
<td>0-1</td>
<td>225600</td>
<td>4187</td>
<td>0.12</td>
<td>115</td>
</tr>
<tr>
<td>1-5</td>
<td>925500</td>
<td>845</td>
<td>0.49</td>
<td>10</td>
</tr>
<tr>
<td>5-10</td>
<td>1198500</td>
<td>457</td>
<td>0.41</td>
<td>10</td>
</tr>
<tr>
<td>10-15</td>
<td>1110100</td>
<td>364</td>
<td>0.48</td>
<td>20</td>
</tr>
<tr>
<td>15-20</td>
<td>1051500</td>
<td>962</td>
<td>0.42</td>
<td>33</td>
</tr>
<tr>
<td>20-25</td>
<td>930500</td>
<td>1081</td>
<td>0.43</td>
<td>51</td>
</tr>
<tr>
<td>25-30</td>
<td>773000</td>
<td>871</td>
<td>0.47</td>
<td>90</td>
</tr>
<tr>
<td>30-35</td>
<td>705800</td>
<td>903</td>
<td>0.35</td>
<td>140</td>
</tr>
<tr>
<td>35-40</td>
<td>754200</td>
<td>1393</td>
<td>0.48</td>
<td>334</td>
</tr>
<tr>
<td>40-45</td>
<td>778300</td>
<td>2461</td>
<td>0.53</td>
<td>811</td>
</tr>
<tr>
<td>45-50</td>
<td>701300</td>
<td>3543</td>
<td>0.56</td>
<td>1570</td>
</tr>
<tr>
<td>50-55</td>
<td>648300</td>
<td>5184</td>
<td>0.52</td>
<td>2501</td>
</tr>
<tr>
<td>55-60</td>
<td>560300</td>
<td>7239</td>
<td>0.54</td>
<td>3881</td>
</tr>
<tr>
<td>60-65</td>
<td>446400</td>
<td>9086</td>
<td>0.54</td>
<td>5170</td>
</tr>
<tr>
<td>65-70</td>
<td>361400</td>
<td>11368</td>
<td>0.54</td>
<td>6753</td>
</tr>
<tr>
<td>70-75</td>
<td>277700</td>
<td>13495</td>
<td>0.54</td>
<td>8472</td>
</tr>
<tr>
<td>75-80</td>
<td>200700</td>
<td>15116</td>
<td>0.53</td>
<td>9843</td>
</tr>
<tr>
<td>80-85</td>
<td>105300</td>
<td>12585</td>
<td>0.54</td>
<td>8556</td>
</tr>
<tr>
<td>85+</td>
<td>55800</td>
<td>11542</td>
<td></td>
<td>7949</td>
</tr>
</tbody>
</table>

Note: The table shows mid-year population and deaths by age and cause for Australia in 1967. The fractions and causes of death are listed for each age group.
3. Multiple Decrement Tables for Sweden and Australia Populations

Two multiple decrement tables have been computed for the Australian population, 1967, and the Swedish population, 1967 [Tables 5 and 6]. The selected causes are cardiovascular diseases (R1), cancer all forms (R2), all accidents (R3), infectious diseases (R4), respiratory diseases (R5), motor vehicle accidents (R6) and all other causes (R7). The crude probability of dying from each specific cause has been computed for every age interval.

In addition, the probability of dying \( q_i \) without reference to cause of death is included in the tables so that the magnitude of the probability \( \hat{q}_{i0} \) for each risk \( R_0 \) relative to the total probability \( \hat{q}_i \) can be determined. For example, the ratio \( \hat{q}_{i0}/\hat{q}_i \) will give the proportionate mortality due to a specific risk \( R_0 \).

It may be noted that if \( \hat{q}_{i0} \) is computed for every risk, then the sum of the probabilities \( \hat{q}_{i0} \) overall possible risks \( R_0 \) will be equal to \( \hat{q}_i \) [c.f., equation (2.10)]. The sum of \( \hat{q}_{i0} \) over only selected risks is less than \( \hat{q}_i \).

For the purpose of testing for significance between the probabilities or making other statistical inferences, the standard deviations of the probabilities are also included in the tables.
Table 5
Multiple Decrement Table for Selected Causes of Death and the Standard Error of the Crude Probability of Dying

Sweden population, 1967

<table>
<thead>
<tr>
<th>Age Interval (in years) ( (x_i, x_{i+1}) )</th>
<th>Probability of dying in interval ( (x_i, x_{i+1}) )</th>
<th>Cardiovascular Diseases ( R_1 )</th>
<th>Cancer All Forms ( R_2 )</th>
<th>All Accidents ( R_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_i ) ( e^a q_i )</td>
<td>( \hat{q}<em>{i1} ) ( \hat{q}</em>{i11} )</td>
<td>( \hat{q}<em>{i2} ) ( \hat{q}</em>{i12} )</td>
<td>( \hat{q}<em>{i3} ) ( \hat{q}</em>{i13} )</td>
<td></td>
</tr>
<tr>
<td>(1) ( (2) ) ( (3) ) ( (4) ) ( (5) ) ( (6) ) ( (7) ) ( (8) ) ( (9) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Age Interval (in years) ( (x_i, x_{i+1}) )</th>
<th>Crude Probability of Dying in Interval ( (x_i, x_{i+1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i, x_{i+1} )</td>
<td>( q_i ) ( e^a q_i )</td>
</tr>
<tr>
<td>0-1</td>
<td>0.01275</td>
</tr>
<tr>
<td>1-5</td>
<td>0.00212</td>
</tr>
<tr>
<td>5-10</td>
<td>0.00164</td>
</tr>
<tr>
<td>10-15</td>
<td>0.00138</td>
</tr>
<tr>
<td>15-20</td>
<td>0.00270</td>
</tr>
<tr>
<td>20-25</td>
<td>0.00366</td>
</tr>
<tr>
<td>25-30</td>
<td>0.00465</td>
</tr>
<tr>
<td>30-35</td>
<td>0.00579</td>
</tr>
<tr>
<td>35-40</td>
<td>0.00735</td>
</tr>
<tr>
<td>40-45</td>
<td>0.01136</td>
</tr>
<tr>
<td>45-50</td>
<td>0.01691</td>
</tr>
<tr>
<td>50-55</td>
<td>0.02607</td>
</tr>
<tr>
<td>55-60</td>
<td>0.04090</td>
</tr>
<tr>
<td>60-65</td>
<td>0.06708</td>
</tr>
<tr>
<td>65-70</td>
<td>0.11131</td>
</tr>
<tr>
<td>70-75</td>
<td>0.18111</td>
</tr>
<tr>
<td>75-80</td>
<td>0.29871</td>
</tr>
<tr>
<td>80-85</td>
<td>0.43982</td>
</tr>
</tbody>
</table>
Table 5 (con't)
Multiple Decrement Table for Selected Causes of Death and the Standard Error of the Crude Probability of Dying
Swedish population, 1967

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Infectious Diseases</th>
<th>Respiratory Diseases</th>
<th>Motor Vehicle Accidents</th>
<th>All Other Causes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(xi, xi+1)</td>
<td>Qt4</td>
<td>Qt5</td>
<td>q16</td>
<td>q17</td>
</tr>
<tr>
<td>0-1</td>
<td>0.001</td>
<td>0.00031</td>
<td>0.004</td>
<td>0.00060</td>
</tr>
<tr>
<td>1-5</td>
<td>0.001</td>
<td>0.00032</td>
<td>0.003</td>
<td>0.00052</td>
</tr>
<tr>
<td>5-10</td>
<td>0.000</td>
<td>0.00017</td>
<td>0.001</td>
<td>0.00025</td>
</tr>
<tr>
<td>10-15</td>
<td>0.000</td>
<td>0.00013</td>
<td>0.001</td>
<td>0.00025</td>
</tr>
<tr>
<td>15-20</td>
<td>0.000</td>
<td>0.00024</td>
<td>0.001</td>
<td>0.00028</td>
</tr>
<tr>
<td>20-25</td>
<td>0.000</td>
<td>0.00019</td>
<td>0.001</td>
<td>0.00024</td>
</tr>
<tr>
<td>25-30</td>
<td>0.000</td>
<td>0.00020</td>
<td>0.001</td>
<td>0.00026</td>
</tr>
<tr>
<td>30-35</td>
<td>0.001</td>
<td>0.00027</td>
<td>0.001</td>
<td>0.00032</td>
</tr>
<tr>
<td>35-40</td>
<td>0.001</td>
<td>0.00039</td>
<td>0.002</td>
<td>0.00043</td>
</tr>
<tr>
<td>40-45</td>
<td>0.002</td>
<td>0.00046</td>
<td>0.003</td>
<td>0.00061</td>
</tr>
<tr>
<td>45-50</td>
<td>0.002</td>
<td>0.00046</td>
<td>0.005</td>
<td>0.00069</td>
</tr>
<tr>
<td>50-55</td>
<td>0.003</td>
<td>0.00054</td>
<td>0.007</td>
<td>0.00084</td>
</tr>
<tr>
<td>55-60</td>
<td>0.005</td>
<td>0.00074</td>
<td>0.011</td>
<td>0.00105</td>
</tr>
<tr>
<td>60-65</td>
<td>0.005</td>
<td>0.00080</td>
<td>0.012</td>
<td>0.00105</td>
</tr>
<tr>
<td>65-70</td>
<td>0.009</td>
<td>0.00108</td>
<td>0.014</td>
<td>0.00124</td>
</tr>
<tr>
<td>70-75</td>
<td>0.012</td>
<td>0.00134</td>
<td>0.016</td>
<td>0.00144</td>
</tr>
<tr>
<td>75-80</td>
<td>0.015</td>
<td>0.00187</td>
<td>0.022</td>
<td>0.00244</td>
</tr>
<tr>
<td>80-85</td>
<td>0.025</td>
<td>0.00294</td>
<td>0.041</td>
<td>0.00167</td>
</tr>
</tbody>
</table>

Note: The table continues with similar data entries for each age interval.
Table 6
Multiple Decrement Table for Selected Causes of Death and the Standard Error of the Crude Probability of Dying

Australia Population, 1967

<table>
<thead>
<tr>
<th>Age Interval (in years) (x_i, x_{i+1})</th>
<th>Probability of dying in interval (q_{i1}, q_{i+1})</th>
<th>Crude Probability of Dying in Interval (R_1, R_2, R_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0-1)</td>
<td>(0.00001826, 0.0000280)</td>
<td>(0.000019, 0.000019)</td>
</tr>
<tr>
<td>(1-5)</td>
<td>(0.0000365, 0.0000125)</td>
<td>(0.000014, 0.0000037)</td>
</tr>
<tr>
<td>(5-10)</td>
<td>(0.0000190, 0.000089)</td>
<td>(0.000014, 0.000040)</td>
</tr>
<tr>
<td>(10-15)</td>
<td>(0.0000164, 0.000086)</td>
<td>(0.000020, 0.000034)</td>
</tr>
<tr>
<td>(15-20)</td>
<td>(0.0000456, 0.0000147)</td>
<td>(0.000027, 0.000042)</td>
</tr>
<tr>
<td>(20-25)</td>
<td>(0.0000579, 0.0000176)</td>
<td>(0.000038, 0.000047)</td>
</tr>
<tr>
<td>(25-30)</td>
<td>(0.0000562, 0.0000190)</td>
<td>(0.000061, 0.000059)</td>
</tr>
<tr>
<td>(30-35)</td>
<td>(0.0000637, 0.0000211)</td>
<td>(0.000083, 0.000083)</td>
</tr>
<tr>
<td>(35-40)</td>
<td>(0.0000919, 0.0000245)</td>
<td>(0.000120, 0.000106)</td>
</tr>
<tr>
<td>(40-45)</td>
<td>(0.0001569, 0.0000314)</td>
<td>(0.000181, 0.000144)</td>
</tr>
<tr>
<td>(45-50)</td>
<td>(0.0002498, 0.0000414)</td>
<td>(0.000278, 0.000193)</td>
</tr>
<tr>
<td>(50-55)</td>
<td>(0.0003923, 0.0000534)</td>
<td>(0.000375, 0.000268)</td>
</tr>
<tr>
<td>(55-60)</td>
<td>(0.0006274, 0.0000714)</td>
<td>(0.000531, 0.000347)</td>
</tr>
<tr>
<td>(60-65)</td>
<td>(0.0009722, 0.0000969)</td>
<td>(0.000748, 0.000475)</td>
</tr>
<tr>
<td>(65-70)</td>
<td>(0.14667, 0.001271)</td>
<td>(0.001013, 0.000636)</td>
</tr>
<tr>
<td>(70-75)</td>
<td>(0.21855, 0.001663)</td>
<td>(0.001385, 0.000761)</td>
</tr>
<tr>
<td>(75-80)</td>
<td>(0.31995, 0.002146)</td>
<td>(0.001869, 0.000964)</td>
</tr>
<tr>
<td>(80-85)</td>
<td>(0.40273, 0.003045)</td>
<td>(0.002844, 0.001379)</td>
</tr>
</tbody>
</table>
### Table 6 (con't)

Multiple Decrement Table for Selected Causes of Death and the Standard Error of the Crude Probability of Dying

*Australia Population, 1967*

<table>
<thead>
<tr>
<th>Age Interval (in years) ( (x_i, x_{i+1}) )</th>
<th>Infectious Diseases ( R_4 )</th>
<th>Respiratory Diseases ( R_5 )</th>
<th>Motor Vehicle Accidents ( R_6 )</th>
<th>All Other Causes ( R_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{q}_{x_i} )</td>
<td>( s_{x_i} )</td>
<td>( \hat{q}<em>{x</em>{i+1}} )</td>
<td>( s_{x_{i+1}} )</td>
<td>( \hat{q}_{x_i} )</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
<td>-------------------------------</td>
<td>-----------------------------</td>
<td>-------------------------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>0-1</td>
<td>0.0002</td>
<td>0.000033</td>
<td>0.0017</td>
<td>0.000987</td>
</tr>
<tr>
<td>1-5</td>
<td>0.0002</td>
<td>0.000029</td>
<td>0.0006</td>
<td>0.000046</td>
</tr>
<tr>
<td>5-10</td>
<td>0.0001</td>
<td>0.000015</td>
<td>0.0001</td>
<td>0.000024</td>
</tr>
<tr>
<td>10-15</td>
<td>0.0000</td>
<td>0.000012</td>
<td>0.0000</td>
<td>0.000013</td>
</tr>
<tr>
<td>15-20</td>
<td>0.0000</td>
<td>0.000015</td>
<td>0.0001</td>
<td>0.000023</td>
</tr>
<tr>
<td>20-25</td>
<td>0.0000</td>
<td>0.000014</td>
<td>0.0001</td>
<td>0.000025</td>
</tr>
<tr>
<td>25-30</td>
<td>0.0001</td>
<td>0.000018</td>
<td>0.0002</td>
<td>0.000032</td>
</tr>
<tr>
<td>30-35</td>
<td>0.0000</td>
<td>0.000017</td>
<td>0.0002</td>
<td>0.000034</td>
</tr>
<tr>
<td>35-40</td>
<td>0.0001</td>
<td>0.000024</td>
<td>0.0003</td>
<td>0.000044</td>
</tr>
<tr>
<td>40-45</td>
<td>0.0002</td>
<td>0.000033</td>
<td>0.0016</td>
<td>0.000060</td>
</tr>
<tr>
<td>45-50</td>
<td>0.0002</td>
<td>0.000035</td>
<td>0.0009</td>
<td>0.000079</td>
</tr>
<tr>
<td>50-55</td>
<td>0.0003</td>
<td>0.000045</td>
<td>0.0014</td>
<td>0.000103</td>
</tr>
<tr>
<td>55-60</td>
<td>0.0005</td>
<td>0.000067</td>
<td>0.0029</td>
<td>0.000158</td>
</tr>
<tr>
<td>60-65</td>
<td>0.0006</td>
<td>0.000080</td>
<td>0.0052</td>
<td>0.000235</td>
</tr>
<tr>
<td>65-70</td>
<td>0.0010</td>
<td>0.000111</td>
<td>0.0091</td>
<td>0.000341</td>
</tr>
<tr>
<td>70-75</td>
<td>0.0016</td>
<td>0.000159</td>
<td>0.0146</td>
<td>0.000483</td>
</tr>
<tr>
<td>75-80</td>
<td>0.0012</td>
<td>0.000162</td>
<td>0.0237</td>
<td>0.000700</td>
</tr>
<tr>
<td>80-85</td>
<td>0.0011</td>
<td>0.000200</td>
<td>0.0376</td>
<td>0.001161</td>
</tr>
</tbody>
</table>
4. Interpretation of a Multiple Decrement Table

A multiple decrement table, such as those presented in Tables 5 and 6, can serve many useful purposes. Significant points include the following:

1. Each probability \(^\wedge\) represents a measure of risk of dying from a specific cause to which a person is subject in a real population where other competing risks are also acting. For example, in Table 5, Column (4), age interval (60,65), we found \(^\wedge \) = .0315. This figure suggests that if the forces of mortality operating in the Swedish population, 1967, prevail, the probability is over three percent that a person of 60 years of age will die from cardiovascular disease within five years.

2. The entire array of probabilities \(^\wedge\) over all the age groups gives a profile of risk of dying from a specific cause during a person's lifetime. Thus the risk of dying from cardiovascular disease is negligible among young people, but increases with advancement of age. According to the Swedish, 1967, experiences, these diseases are the most serious cause of death for persons beyond age 50, and the chance is better than one in four (.2771) that a person of age 60 will die from cardiovascular disease in the following five years despite competition from other causes. Below age 40, however, cardiovascular disease is negligible as a cause of death, and between ages 1 and 10, the risk of death from these diseases is almost nonexistent. Alternatively, the risk of death from cancer all forms is more evenly spread over the age intervals but is less evenly spread than the risk of death from all accidents.

3. When viewed across several causes of death, the multiple decrement table shows relative risk of death either for a specific age group or for the entire life span. It is evident from these tables that among the leading causes of death, cardiovascular disease is a dominate cause, cancer all forms runs a poor second, while all accidents is a distant third. However, in age 40-50
cancer all forms is the most menacing disease in Sweden in 1967 and is a significant cause of death even below age 40. Within the latter age bracket, however, all accidents assume the leading role as cause of death.

4. Each probability $\hat{q}_i$, when expressed in terms of the probability of dying $q_i$, gives the proportionate mortality for each cause. This in turn provides the information as to what proportion of mortality in each age group may be attributed to specific causes.

5. A comprehensive comparison of cause specific mortality experience may be made among different countries, or of a country over time. Between Sweden and Australia, for example, the general mortality pattern is similar, but details vary. According to 1967 experience, the Australian population is subject to higher risks of death than the Swedish population in almost every age group and for each cause considered in this example. The only exceptions occur in the very old age brackets for a few causes. Beyond age 70 cancer all forms is a more eminent cause in Sweden than in Australia. A similar statement can be made for respiratory disease beyond age 80, and infectious diseases beyond age 75, although the magnitude of the probabilities for the latter case is quite small. It may also be noted that, while in Sweden, 1967, cardiovascular disease was the most serious cause of death from age 50 on, in Australia these diseases assume this role beginning in the early 30's.

6. Statistical inference about these rates can be readily made with the aid of the standard deviations listed. For example, hypotheses can be tested regarding the probability of dying from a specific cause between Sweden and Australia. Is the probability of dying from cardiovascular disease for age group (45,50) greater in Australia than Sweden? To answer this question, we compute the critical ratio (cf., Chapter 4).
which has a normal distribution with a mean zero and a variance one, if in fact \( Q_{45,1}^{(A)} = Q_{45,1}^{(S)} \). The numerical value is

\[
Z = \frac{.0111 - .0044}{\sqrt{(.000278)^2 + (.000200)^2}} = 19.6
\]

which is highly significant, as the corresponding probability is less than 1 in 10,000. In other words, if the probability of dying from cardiovascular diseases in Australia was equal to that in Sweden for a person of age 45-50, then the chances are less than 1 in 10,000 that a difference as great or greater than the one observed would occur. Based on the above findings, we conclude that cardiovascular diseases were a more serious cause of death in Australia than it was in Sweden for the age group under consideration.

7. Caution should be observed in comparing the crude probabilities of dying from different causes in the same population and the same age group. In a particular age group, various causes are competing with one another for the life of an individual and the estimated probabilities are statistically dependent. Between any two probabilities (say, \( \hat{Q}_{i1} \) and \( \hat{Q}_{i2} \)) there is a co-variance, which must be taken into account in making inference about these probabilities. The co-variance between \( \hat{Q}_{i1} \) and \( \hat{Q}_{i2} \), for example, is given by

\[
\text{Cov}(\hat{Q}_{i1}, \hat{Q}_{i2}) = -\frac{1}{n_{i1}} \hat{Q}_{i1} \hat{Q}_{i2} = -\frac{1}{n_{i1}} \hat{Q}_{i1}^2 \hat{Q}_{i2}^2. 
\]
A similar formula holds for any two probabilities. How the covariance can be incorporated in statistical inference is demonstrated below. Suppose we want to compare two causes $R_1$: cardiovascular diseases and $R_2$: cancer all forms for people of age 55-60 in the Sweden population, 1967. From column (4) and (6), we found $\hat{Q}_{55,1} = .0167$ and $\hat{Q}_{55,2} = .0125$, with a difference

$$\hat{Q}_{55,1} - \hat{Q}_{55,2} = .0042 \quad (4.3)$$

Is this difference significantly greater than zero or can it be explained by chance? Here we are testing the hypothesis that $Q_{55,1} = Q_{55,2}$ against the alternative hypothesis that $Q_{55,1} > Q_{55,2}$. To test the hypothesis, we express the difference $\hat{Q}_{55,1} - \hat{Q}_{55,2}$ in terms of its standard deviation, which is given by

$$\text{S.D.}(\hat{Q}_{55,1} - \hat{Q}_{55,2}) = \sqrt{\frac{\hat{Q}_{55,1}^2}{\hat{Q}_{55,1}^2} + \frac{\hat{Q}_{55,2}^2}{\hat{Q}_{55,2}^2} - 2 \text{Cov}(\hat{Q}_{55,1}, \hat{Q}_{55,2})} \quad (4.4)$$

where the covariance can be computed as follows:

$$\text{Cov}(\hat{Q}_{55,1}, \hat{Q}_{55,2}) = -\frac{1}{\hat{Q}_{55,1} \hat{Q}_{55,2}}$$

$$\text{Cov}(\hat{Q}_{55,1}, \hat{Q}_{55,2}) = -.000000002$$

which is a very small number. Now the standard deviation can be computed

$$\text{S.D.}(\hat{Q}_{55,1} - \hat{Q}_{55,2}) = \sqrt{(.000397)^2 + (.000344)^2 + 2(.000000002)}$$

$$= 0.00053$$
The statistic used to test the hypothesis is again the normal deviate

\[ Z = \frac{\hat{q}_{55,1} - \hat{q}_{55,2}}{\text{S.D.}(\hat{q}_{55,1} - \hat{q}_{55,2})} \]  

(4.5)

and compare the numerical value of \( Z \) with the standard normal distribution.

In this case we have

\[ Z = \frac{.0167 - .0125}{.00053} = 7.9 \]

which is highly significant. Thus according to Sweden 1967 experience, the probability of dying from cardiovascular disease is greater than cancer all forms for age interval (55,60).

In conclusion, the multiple decrement table presents a mortality profile over ages and causes of death for a population under study. It shows the relative, as well as the absolute, importance of various diseases and their variation over age and sex. With the information provided in the table, one can easily detect and determine the area of concern, the degree of seriousness of various diseases, and the type and amount of medical care and health services needed by persons in different age and sex categories.
CHAPTER 8

THE LIFE TABLE WHEN A PARTICULAR CAUSE IS ELIMINATED

1. Introduction

In Appendix III on competing risks, several types of probabilities of dying with respect to a particular cause have been discussed. Corresponding to each of these probabilities, a life table may be constructed to serve a specific purpose using the probability in question in place of \( q_i \). The procedure of construction is exactly the same as that of an ordinary life table described in Chapter 4, though the columns have different meanings. A life table derived from \( \hat{q}_{i,1} \), the probability of dying when a cause \( R_1 \) (e.g., cardiovascular-renal disease) is eliminated as a cause of death, for example, may be used to evaluate the effect of the cardiovascular-renal diseases on the longevity of a human population in terms of the expectation of life or chance of survival. Generally, the event involved need not be survival or death and the subject is not limited to human beings. In a study of the effect of divorce on the longevity of marriage, for example, the event is the dissolution of marriage. If divorce \((R_1)\) had been removed as a cause, how long is a marriage expected to last before death occurred to either one of the spouses? Application to other problems is possible wherever the concept of competing risks applies. We shall in this chapter describe two tables: (1) the life table when a specific cause is eliminated, and (2) the life table when a specific risk is the only risk operating. Empirical data will be used for illustration.

2. Computation of the Net Probability, \( \hat{q}_{i,1} \)

The life table in this chapter is derived from \( q_{i,1} \), the net probability of dying when a particular cause \( R_1 \) is eliminated. This is one of the most important
applications of the competing risks theory. Such a table may be constructed either for a cohort population or for a current population. In either case, the basic formula is (cf., Equation (2.29) in Appendix III)

\[ q_{i,1} = (q_i - Q_{i1})(1 + \frac{1}{2} Q_{i1}) \]  

(2.1)

where \( q_i \) is the probability of dying in interval \( (x_i, x_{i+1}) \) and \( Q_{i1} \) is the crude probability of dying from \( R_1 \) during the same interval in the presence of other competing risks. To avoid repetition, only the life table derived from mortality data of a current population will be discussed.

Let us consider, as an example, cardiovascular-renal diseases and the effect of their presence on the probability of dying and the expectation of life. For a typical age interval \( (x_i, x_{i+1}) \), with \( x_{i+1} - x_i = n_i \) being the interval length, let \( D_i \) be the number of deaths occurring in age interval \( (x_i, x_{i+1}) \) during a calendar year, among them \( D_{i1} \) dying from \( R_1 \). Let \( p_i \) be the corresponding mid-year population, and \( a_i \) the fraction of last age interval of life. The age-specific death rate is computed as before from

\[ M_i = \frac{D_i}{p_i} \]  

(2.2)

and death rate specific for cause \( R_1 \) (CVR diseases) is computed from

\[ M_{i1} = \frac{D_{i1}}{p_i} . \]  

(2.3)

Using the results in preceding chapters, we have the estimate of the probability [cf., Equation (4.3), Chapter 3]

\[ \hat{q}_i = \frac{n_i M_i}{1 + (1-a_i) n_i M_i} \]  

(2.4)
and of the crude probability [cf., equation (2.9), Chapter 7]

\[
\hat{Q}_{i1} = \frac{n_i M_{i1}}{1 + \left(1-a_i\right) \frac{n_i M_i}{n_i M_{i1}}}.
\]  

(2.5)

Substituting (2.4) and (2.5) in (2.1) yields the required estimate of the probability \(q_{i,1}\):

\[
\hat{q}_{i,1} = \left(\hat{q}_i - \hat{Q}_{i1}\right) \left(1 + \frac{1}{2} \hat{Q}_{i1}\right).
\]  

(2.6)

With reference to formulas (2.1) through (2.6) above, computation of \(\hat{q}_{i,1}\) is summarized in Table 1. For each age interval \((x_i, x_{i+1})\), the data required are midyear population \(P_i\) (Column 2), number of deaths from all causes \(D_i\) (Column 3), and number of deaths from the cause under study (in this case, cardiovascular-renal diseases) \(D_{i1}\) (Column 4). These figures, which are available in population and vital statistics publications, are used to compute death rate \(M_i\) (Column 5) and cause-specific death rate \(M_{i1}\) (Column 6). The fraction of last age interval of life, \(a_i\), in Column (7) is given in Appendix V. Using formulas (2.4), (2.5) and (2.6) the probabilities \(\hat{q}_i\), \(\hat{Q}_{i1}\), and finally \(\hat{q}_{i,1}\), are computed and recorded in Columns (8), (9), and (10), respectively.

For age interval \((0, 1)\), for example, we have the midyear population \(P_0 = 1,794,784\), the number of deaths \(D_0 = 48,063\), deaths from cardiovascular diseases \(D_{01} = 228\), and \(a_0 = .10\). With these values, we compute the death rate from all causes using formula (2):

\[
M_0 = \frac{D_0}{P_0} = \frac{48,063}{1,794,784} = .026779
\]  

(2.2a)

or 26.78 per 1,000, and the death rate from cardiovascular-renal diseases using formula (2.3):
The probability of dying, \( \hat{q}_0 \), is computed from formula (2.4) which is the same as in the ordinary life table in Chapter 3, namely

\[
\hat{q}_0 = \frac{M_0}{1 + (1-a_0) M_0} = \frac{.026779}{1 + (.90) .026779} = .02615
\]

(2.4a)

and the crude probability of dying from cardiovascular-renal diseases is computed from

\[
\hat{q}_{01} = \frac{M_{01}}{1 + (1-a_0) M_0} = \frac{.000127}{1 + (.90) .026779} = .000124
\]

(2.5a)

and finally, the net probability \( q_{0.1} \)

\[
\hat{q}_{0.1} = (\hat{q}_0 - \hat{q}_{01})(1 + \frac{1}{2} \hat{q}_{01}) = (.02615 - .000124)(1 + \frac{1}{2} .000124) = .026028
\]

(2.6a)

For age interval \((1, 5)\), \( n_1 = 5-1 = 4 \), the rates and probabilities are successively computed as follows:

\[
M_1 = \frac{D_1}{P_1} = \frac{7,409}{7,063,044} = .001049 \text{ or } 1.049 \text{ per 1,000}
\]

(2.2a)
Table 1.

Computation of the net probability of dying, \( q_{i,1} \), when cardiovascular-renal (CVR) diseases (R1) are eliminated as a cause of death, white males, United States, 1960.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Mid-year Populations (a)</th>
<th>Deaths from all causes (b)</th>
<th>Deaths from Cardiovascular renal diseases (c,d)</th>
<th>Death rate from all causes (M)</th>
<th>Death rate from CVR (( M_{il} ))</th>
<th>Fraction of last Age Interval of Life (( \hat{a}_i ))</th>
<th>Probability of dying (( \hat{q}_i ))</th>
<th>Crude Probability of dying from CVR (( \hat{q}_{i}^{\text{CVR}} ))</th>
<th>Net Probability of dying when CVR is eliminated (( q_{i,1} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>1794784</td>
<td>48063</td>
<td>228</td>
<td>0.026779</td>
<td>0.00127</td>
<td>0.10</td>
<td>0.02615</td>
<td>0.000124</td>
<td>0.02603</td>
</tr>
<tr>
<td>1-5</td>
<td>7063044</td>
<td>7409</td>
<td>153</td>
<td>0.001049</td>
<td>0.00022</td>
<td>0.39</td>
<td>0.00419</td>
<td>0.00088</td>
<td>0.00410</td>
</tr>
<tr>
<td>5-10</td>
<td>8191158</td>
<td>4408</td>
<td>177</td>
<td>0.000538</td>
<td>0.00024</td>
<td>0.46</td>
<td>0.00269</td>
<td>0.00108</td>
<td>0.00258</td>
</tr>
<tr>
<td>10-15</td>
<td>7488562</td>
<td>3847</td>
<td>208</td>
<td>0.000514</td>
<td>0.00028</td>
<td>0.54</td>
<td>0.00257</td>
<td>0.00139</td>
<td>0.00243</td>
</tr>
<tr>
<td>15-20</td>
<td>5893946</td>
<td>7308</td>
<td>355</td>
<td>0.001240</td>
<td>0.00060</td>
<td>0.57</td>
<td>0.00618</td>
<td>0.00300</td>
<td>0.00588</td>
</tr>
<tr>
<td>20-25</td>
<td>4654740</td>
<td>7755</td>
<td>481</td>
<td>0.001665</td>
<td>0.00103</td>
<td>0.49</td>
<td>0.00829</td>
<td>0.000514</td>
<td>0.00778</td>
</tr>
<tr>
<td>25-30</td>
<td>4725480</td>
<td>7182</td>
<td>768</td>
<td>0.001520</td>
<td>0.00163</td>
<td>0.50</td>
<td>0.00757</td>
<td>0.000810</td>
<td>0.00676</td>
</tr>
<tr>
<td>30-35</td>
<td>5216424</td>
<td>9039</td>
<td>1808</td>
<td>0.001733</td>
<td>0.000347</td>
<td>0.52</td>
<td>0.00863</td>
<td>0.001726</td>
<td>0.00851</td>
</tr>
<tr>
<td>35-40</td>
<td>5616528</td>
<td>13803</td>
<td>4444</td>
<td>0.002527</td>
<td>0.000814</td>
<td>0.54</td>
<td>0.01256</td>
<td>0.004045</td>
<td>0.00854</td>
</tr>
<tr>
<td>40-45</td>
<td>5094821</td>
<td>21336</td>
<td>9125</td>
<td>0.004188</td>
<td>0.001791</td>
<td>0.54</td>
<td>0.02074</td>
<td>0.008870</td>
<td>0.01192</td>
</tr>
<tr>
<td>45-50</td>
<td>4850486</td>
<td>32447</td>
<td>16796</td>
<td>0.007061</td>
<td>0.003463</td>
<td>0.54</td>
<td>0.03474</td>
<td>0.017037</td>
<td>0.01785</td>
</tr>
<tr>
<td>50-55</td>
<td>4314976</td>
<td>50716</td>
<td>26812</td>
<td>0.011753</td>
<td>0.006214</td>
<td>0.53</td>
<td>0.05719</td>
<td>0.030233</td>
<td>0.02737</td>
</tr>
<tr>
<td>55-60</td>
<td>3774623</td>
<td>6540</td>
<td>36907</td>
<td>0.017628</td>
<td>0.007978</td>
<td>0.52</td>
<td>0.08456</td>
<td>0.046904</td>
<td>0.03858</td>
</tr>
<tr>
<td>60-65</td>
<td>3100043</td>
<td>85890</td>
<td>49649</td>
<td>0.027706</td>
<td>0.016016</td>
<td>0.52</td>
<td>0.12989</td>
<td>0.075085</td>
<td>0.05702</td>
</tr>
<tr>
<td>65-70</td>
<td>2637044</td>
<td>108726</td>
<td>65069</td>
<td>0.041230</td>
<td>0.024880</td>
<td>0.52</td>
<td>0.18759</td>
<td>0.113198</td>
<td>0.07908</td>
</tr>
<tr>
<td>70-75</td>
<td>1972947</td>
<td>119269</td>
<td>75371</td>
<td>0.060652</td>
<td>0.038202</td>
<td>0.51</td>
<td>0.26327</td>
<td>0.166370</td>
<td>0.10636</td>
</tr>
<tr>
<td>75-80</td>
<td>1214577</td>
<td>109193</td>
<td>73057</td>
<td>0.089902</td>
<td>0.060150</td>
<td>0.51</td>
<td>0.36837</td>
<td>0.246655</td>
<td>0.14106</td>
</tr>
<tr>
<td>80-85</td>
<td>5912511</td>
<td>83885</td>
<td>58713</td>
<td>0.141877</td>
<td>0.099303</td>
<td>0.48</td>
<td>0.51822</td>
<td>0.362166</td>
<td>0.19679</td>
</tr>
<tr>
<td>85-90</td>
<td>2355661</td>
<td>4902</td>
<td>3133</td>
<td>0.210141</td>
<td>0.153388</td>
<td>0.45</td>
<td>0.66589</td>
<td>0.486555</td>
<td>0.25627</td>
</tr>
<tr>
<td>90-95</td>
<td>56704</td>
<td>18253</td>
<td>13604</td>
<td>0.321900</td>
<td>0.239913</td>
<td>0.41</td>
<td>0.82555</td>
<td>0.615285</td>
<td>0.35901</td>
</tr>
<tr>
<td>95+</td>
<td>12333</td>
<td>4219</td>
<td>3136</td>
<td>0.342090</td>
<td>0.254277</td>
<td>1.00</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>TOTAL</td>
<td>78347769</td>
<td>860857d</td>
<td>473640d</td>
<td>0.010988</td>
<td>0.006045</td>
<td>1.00</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
</tbody>
</table>


d. Including those age not stated, 267 and 106, respectively.
\[ M_{11} = \frac{D_{11}}{p_1} = \frac{153}{7,063,044} \]

\[ = 0.000022 \text{ or } 0.022 \text{ per 1,000} \]  

(2.3a)

\[ \hat{q}_1 = \frac{4M_{11}}{1 + (1 - a_1) 4M_{11}} = \frac{4(0.001049)}{1 + (1 - 0.39) 4(0.001049)} \]

\[ = 0.00419 \]  

(2.4a)

\[ \hat{q}_{11} = \frac{4M_{11}}{1 + (1 - a_1) 4M_{11}} = \frac{4(0.000022)}{1 + (1 - 0.39) 4(0.001049)} \]

\[ = 0.000088 \]  

(2.5a)

and

\[ \hat{q}_{1.1} = \hat{q}_1 - \hat{q}_{11} (1 + \frac{1}{2} \hat{q}_{11}) \]

\[ = (0.00419 - 0.000088) (1 + \frac{1}{2} \cdot 0.000088) = 0.004102. \]  

(2.6a)

### 3. Construction of the Life Table

When all the values of \( \hat{q}_{i.1} \) are computed, the columns in the life table can be obtained following the procedure described in Chapter 3. Beginning with a radix \( l_{0.1} = 100,000 \), we compute the number of deaths in \((0, 1)^*\),

\[ d_{0.1} = l_{0.1} q_{0.1} \]

\[ = 100,000 \times 0.02603 = 2603 \]  

(3.1)

the number living at age 1

\[ l_{1.1} = l_{0.1} - d_{0.1} \]

\[ = 100,000 - 2603 = 97397 \]  

(3.2)

*An notation \( .1 \) is added in the subscript of \( l_{i.1}, d_{i.1}, T_{i.1}, \text{ and } e_{i.1} \) to indicate that \( R_i \) is eliminated.*
and the number of years lived in \((0, 1)\),

\[ L_{0.1} = (l_{0.1} - d_{0.1}) + a_0 d_{0.1} \]

\[ = 97397 + .1 \times 2603 = 97657 \]  \hspace{1cm} (3.3)

Other figures in these columns for the subsequent age intervals (except for the last interval) can be computed in exactly the same way.

The computations for the last age interval (e.g., 95 and over) have been described in Chapter 3 [cf., Equations (3.10) to (3.12) in Chapter 3]. For easy reference, they are restated below. The number living at age 95, \(l_{95.1} = 21564\), is the survivors of interval \((90, 95)\). The expectation of life \(e_{95.1}\) is computed directly from the death rate from causes other than cardiovascular-renal disease in the current population. Using the inverse relationship between the expectation and the death rate (cf. equation (3.7) in Chapter 3)

\[ \hat{e}_{95.1} = \frac{p_{95}}{d_{95} - d_{95.1}} = \frac{12333}{4219 - 3136} \]

\[ = 11.3878 \text{ years} \]  \hspace{1cm} (3.4)

Since

\[ \hat{e}_{95.1} = \frac{t_{95.1}}{l_{95.1}} \]  \hspace{1cm} (3.5)

we have

\[ t_{95.1} = l_{95.1} \hat{e}_{95.1} \]

\[ = 21564 \times 11.3878 \]

\[ = 245567 \].
The remaining quantities in the last age interval may be derived from the obvious relationships, thus

\[ L_{95.1} = T_{95.1} = 245,567 \]  \hspace{1cm} (3.6)

\[ d_{95.1} = L_{95.1} = 21,564 \]  \hspace{1cm} (3.7)

and

\[ q_{95.1} = 1.00000 \]  \hspace{1cm} (3.8)

With \( L_{95.1} \) and all other \( L_{i.1} \) determined, we proceed to compute \( T_{i.1} \) from

\[ T_{i.1} = L_{i.1} + \ldots + L_{95.1} \]  \hspace{1cm} (3.9)

For convenience, \( T_{i.1} \) are computed successively from the highest age group, beginning with \( T_{95.1} = 245,567 \). For age 90, \( T_{90.1} \) is computed from

\[ T_{90.1} = L_{90.1} + T_{95.1} \]

\[ = 132,580 + 245,567 = 378,147 \]

and \( T_{85.1} \) from

\[ T_{85.1} = L_{85.1} + T_{90.1} \]

and so on. In general,

\[ T_{i.1} = L_{i.1} + T_{i+1.1} \]  \hspace{1cm} (3.10)

The expectation of life (except for \( \hat{e}_{95.1} \)) is then obtained from

\[ \hat{e}_{i.1} = \frac{T_{i.1}}{L_{i.1}} \]  \hspace{1cm} (3.11)

for each \( i \). For example,
Table 2.

Abridged Life Table when Cardiovascular Renal Diseases are eliminated as a cause of death for white males, United States, 1960.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval ( (x_i, x_{i+1}) ) ( q_{i.1} )</th>
<th>Number living at age ( x_i ) ( L_{i.1} )</th>
<th>Number dying in interval ( (x_i, x_{i+1}) ) ( d_{i.1} )</th>
<th>Fraction of last age interval of life ( a_i )</th>
<th>Number of years lived in interval ( (x_i, x_{i+1}) ) ( L_{i.1} )</th>
<th>Total number of years lived beyond age ( x_i ) ( T_{i.1} )</th>
<th>Expectation of life at age ( x_i ) ( e_{i.1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i ) to ( x_{i+1} )</td>
<td>( q_{i.1} )</td>
<td>( L_{i.1} )</td>
<td>( d_{i.1} )</td>
<td>( a_i )</td>
<td>( L_{i.1} )</td>
<td>( T_{i.1} )</td>
<td>( e_{i.1} )</td>
</tr>
<tr>
<td>0-1</td>
<td>0.02603</td>
<td>100000</td>
<td>2603</td>
<td>0.10</td>
<td>97657</td>
<td>7894729</td>
<td>78.95</td>
</tr>
<tr>
<td>1-5</td>
<td>0.00410</td>
<td>97397</td>
<td>399</td>
<td>0.03</td>
<td>98614</td>
<td>7797072</td>
<td>77.80</td>
</tr>
<tr>
<td>5-10</td>
<td>0.00258</td>
<td>96998</td>
<td>250</td>
<td>0.26</td>
<td>484315</td>
<td>7408458</td>
<td>76.38</td>
</tr>
<tr>
<td>10-15</td>
<td>0.00243</td>
<td>96748</td>
<td>235</td>
<td>0.54</td>
<td>483199</td>
<td>6924143</td>
<td>71.57</td>
</tr>
<tr>
<td>15-20</td>
<td>0.00588</td>
<td>96513</td>
<td>567</td>
<td>0.57</td>
<td>481346</td>
<td>6440944</td>
<td>66.74</td>
</tr>
<tr>
<td>20-25</td>
<td>0.00778</td>
<td>95946</td>
<td>746</td>
<td>0.49</td>
<td>477828</td>
<td>5959598</td>
<td>62.11</td>
</tr>
<tr>
<td>25-30</td>
<td>0.00676</td>
<td>95200</td>
<td>644</td>
<td>0.50</td>
<td>474390</td>
<td>5481770</td>
<td>57.58</td>
</tr>
<tr>
<td>30-35</td>
<td>0.00691</td>
<td>94556</td>
<td>653</td>
<td>0.52</td>
<td>471213</td>
<td>5007380</td>
<td>52.96</td>
</tr>
<tr>
<td>35-40</td>
<td>0.00854</td>
<td>93903</td>
<td>802</td>
<td>0.54</td>
<td>467670</td>
<td>4536167</td>
<td>48.31</td>
</tr>
<tr>
<td>40-45</td>
<td>0.01192</td>
<td>93101</td>
<td>1110</td>
<td>0.54</td>
<td>462952</td>
<td>4068497</td>
<td>43.70</td>
</tr>
<tr>
<td>45-50</td>
<td>0.01785</td>
<td>91991</td>
<td>1643</td>
<td>0.54</td>
<td>456178</td>
<td>3605545</td>
<td>39.19</td>
</tr>
<tr>
<td>50-55</td>
<td>0.02737</td>
<td>90349</td>
<td>2473</td>
<td>0.53</td>
<td>445933</td>
<td>3149367</td>
<td>34.86</td>
</tr>
<tr>
<td>55-60</td>
<td>0.03858</td>
<td>87876</td>
<td>3390</td>
<td>0.52</td>
<td>431244</td>
<td>2703434</td>
<td>30.76</td>
</tr>
<tr>
<td>60-65</td>
<td>0.05702</td>
<td>84486</td>
<td>4817</td>
<td>0.52</td>
<td>410869</td>
<td>2272190</td>
<td>26.89</td>
</tr>
<tr>
<td>65-70</td>
<td>0.07908</td>
<td>79669</td>
<td>6300</td>
<td>0.52</td>
<td>383225</td>
<td>1861321</td>
<td>23.36</td>
</tr>
<tr>
<td>70-75</td>
<td>0.10636</td>
<td>73369</td>
<td>7804</td>
<td>0.51</td>
<td>347725</td>
<td>1478096</td>
<td>20.15</td>
</tr>
<tr>
<td>75-80</td>
<td>0.14106</td>
<td>65565</td>
<td>9249</td>
<td>0.51</td>
<td>305165</td>
<td>1130371</td>
<td>17.24</td>
</tr>
<tr>
<td>80-85</td>
<td>0.19679</td>
<td>56316</td>
<td>11082</td>
<td>0.48</td>
<td>252767</td>
<td>825206</td>
<td>14.65</td>
</tr>
<tr>
<td>85-90</td>
<td>0.25627</td>
<td>45234</td>
<td>11592</td>
<td>0.45</td>
<td>194292</td>
<td>572439</td>
<td>12.66</td>
</tr>
<tr>
<td>90-95</td>
<td>0.35901</td>
<td>33642</td>
<td>12078</td>
<td>0.41</td>
<td>132580</td>
<td>378147</td>
<td>11.24</td>
</tr>
<tr>
<td>95+</td>
<td>1.00000</td>
<td>21564</td>
<td>21564</td>
<td></td>
<td>245567</td>
<td>245567</td>
<td>11.39</td>
</tr>
</tbody>
</table>
### Table 3.

Abridged Life Table when Cardiovascular Renal Diseases are Eliminated as a cause of death for white females, United States, 1960.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval ( (x_i, x_{i+1}) )</th>
<th>Number living at age ( x_i )</th>
<th>Number dying in interval ( (x_i, x_{i+1}) )</th>
<th>Fraction of last age interval of life</th>
<th>Number of years lived in interval beyond age</th>
<th>Total number of years lived beyond age</th>
<th>Expectation of life at age ( x_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i ) to ( x_{i+1} )</td>
<td>( a_{i,1} )</td>
<td>( l_{i,1} )</td>
<td>( d_{i,1} )</td>
<td>( a_i )</td>
<td>( l_{i,1} )</td>
<td>( e_{i,1} )</td>
<td></td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
</tr>
<tr>
<td>0 - 1</td>
<td>.01959</td>
<td>100000</td>
<td>1959</td>
<td>.10</td>
<td>98237</td>
<td>8676848</td>
<td>86.77</td>
</tr>
<tr>
<td>1 - 5</td>
<td>.00334</td>
<td>98041</td>
<td>327</td>
<td>.39</td>
<td>391366</td>
<td>8578611</td>
<td>87.50</td>
</tr>
<tr>
<td>5 - 10</td>
<td>.00184</td>
<td>97714</td>
<td>180</td>
<td>.46</td>
<td>488084</td>
<td>8187245</td>
<td>88.77</td>
</tr>
<tr>
<td>10 - 15</td>
<td>.00140</td>
<td>97534</td>
<td>137</td>
<td>.54</td>
<td>487355</td>
<td>7699161</td>
<td>88.94</td>
</tr>
<tr>
<td>15 - 20</td>
<td>.00227</td>
<td>97297</td>
<td>221</td>
<td>.57</td>
<td>486510</td>
<td>7211806</td>
<td>87.26</td>
</tr>
<tr>
<td>20 - 25</td>
<td>.00259</td>
<td>97176</td>
<td>252</td>
<td>.49</td>
<td>485237</td>
<td>6725296</td>
<td>69.21</td>
</tr>
<tr>
<td>25 - 30</td>
<td>.00295</td>
<td>96924</td>
<td>286</td>
<td>.50</td>
<td>483905</td>
<td>6240059</td>
<td>64.38</td>
</tr>
<tr>
<td>30 - 35</td>
<td>.00395</td>
<td>96628</td>
<td>382</td>
<td>.52</td>
<td>482273</td>
<td>5756154</td>
<td>59.56</td>
</tr>
<tr>
<td>35 - 40</td>
<td>.00583</td>
<td>96256</td>
<td>561</td>
<td>.54</td>
<td>479990</td>
<td>5273881</td>
<td>54.70</td>
</tr>
<tr>
<td>40 - 45</td>
<td>.0088</td>
<td>95695</td>
<td>843</td>
<td>.54</td>
<td>476536</td>
<td>4793891</td>
<td>50.10</td>
</tr>
<tr>
<td>45 - 50</td>
<td>.01282</td>
<td>94852</td>
<td>1216</td>
<td>.54</td>
<td>474163</td>
<td>4317355</td>
<td>45.52</td>
</tr>
<tr>
<td>50 - 55</td>
<td>.01775</td>
<td>93636</td>
<td>1662</td>
<td>.53</td>
<td>464274</td>
<td>3845892</td>
<td>41.67</td>
</tr>
<tr>
<td>55 - 60</td>
<td>.02287</td>
<td>92197</td>
<td>2103</td>
<td>.52</td>
<td>454823</td>
<td>3381618</td>
<td>36.77</td>
</tr>
<tr>
<td>60 - 65</td>
<td>.03241</td>
<td>89871</td>
<td>2913</td>
<td>.52</td>
<td>442364</td>
<td>2926795</td>
<td>32.57</td>
</tr>
<tr>
<td>65 - 70</td>
<td>.04416</td>
<td>86958</td>
<td>3840</td>
<td>.54</td>
<td>425874</td>
<td>2484431</td>
<td>28.57</td>
</tr>
<tr>
<td>70 - 75</td>
<td>.06179</td>
<td>83118</td>
<td>5136</td>
<td>.51</td>
<td>403007</td>
<td>2058857</td>
<td>24.77</td>
</tr>
<tr>
<td>75 - 80</td>
<td>.08920</td>
<td>77982</td>
<td>6956</td>
<td>.51</td>
<td>372868</td>
<td>1655850</td>
<td>21.23</td>
</tr>
<tr>
<td>80 - 85</td>
<td>.13492</td>
<td>71026</td>
<td>9583</td>
<td>.48</td>
<td>330214</td>
<td>1282982</td>
<td>18.06</td>
</tr>
<tr>
<td>85 - 90</td>
<td>.19040</td>
<td>61443</td>
<td>11699</td>
<td>.45</td>
<td>275043</td>
<td>952768</td>
<td>15.51</td>
</tr>
<tr>
<td>90 - 95</td>
<td>.29170</td>
<td>49744</td>
<td>14510</td>
<td>.41</td>
<td>205915</td>
<td>677725</td>
<td>13.62</td>
</tr>
<tr>
<td>95 +</td>
<td>1.00000</td>
<td>35234</td>
<td>35234</td>
<td>.40</td>
<td>471810</td>
<td>471810</td>
<td>13.39</td>
</tr>
</tbody>
</table>
This completes the procedure of constructing life tables.

We have used the 1960 US white male population as an example to illustrate the high force of mortality from cardiovascular-renal disease. For purposes of comparison, a table for the US white female 1960 population has also been constructed and is reproduced here.

4. Interpretation of Findings

Cardiovascular-renal (CVR) diseases have caused more deaths in the human population than any other disease. As a group, they are responsible for over 55 percent of all deaths in the United States in recent years. Equally impressive figures, but to a somewhat lesser extent, have been reported in the European countries. To evaluate the impact of these diseases on human longevity, we can compare the mortality and survival experience of the current population with the hypothetical experience of the same population under the condition that would exist if CVR diseases were removed as causes of death. The life table and the theory of competing risks provide the most convenient methods for the analysis of such a problem. In Tables 4 to 6, the probability of dying, the survival probability, and the expectation of life are given with and without the presence of CVR, each reflecting in a different way the effect these diseases have on the mortality of the population in question. A brief discussion on these findings follows.

\[
\hat{e}_{90.1} = \frac{T_{90.1}}{E_{90.1}} = \frac{378,147}{33,642} = 11.24
\]
<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>White males</th>
<th></th>
<th></th>
<th></th>
<th>White females</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$q_i$</td>
<td>$\bar{q}_{i+1}$</td>
<td>$q_i - \bar{q}_{i+1}$</td>
<td>$\frac{q_i - \bar{q}_{i+1}}{q_i}$</td>
<td>$\bar{q}_{i}$</td>
<td>$\bar{q}_{i+1}$</td>
<td>$q_i - \bar{q}_{i+1}$</td>
<td>$\frac{q_i - \bar{q}_{i+1}}{q_i}$</td>
</tr>
<tr>
<td>0-1</td>
<td>.02615</td>
<td>.02603</td>
<td>.00012</td>
<td>.05%</td>
<td>.01967</td>
<td>.01959</td>
<td>.00008</td>
<td>.04%</td>
</tr>
<tr>
<td>1-5</td>
<td>.00419</td>
<td>.00410</td>
<td>.00009</td>
<td>2.1</td>
<td>.00341</td>
<td>.00334</td>
<td>.00007</td>
<td>2.1</td>
</tr>
<tr>
<td>5-10</td>
<td>.00269</td>
<td>.00258</td>
<td>.00011</td>
<td>4.1</td>
<td>.00191</td>
<td>.00184</td>
<td>.00007</td>
<td>3.7</td>
</tr>
<tr>
<td>10-15</td>
<td>.00257</td>
<td>.00243</td>
<td>.00014</td>
<td>5.4</td>
<td>.00154</td>
<td>.00140</td>
<td>.00014</td>
<td>9.1</td>
</tr>
<tr>
<td>15-20</td>
<td>.00618</td>
<td>.00588</td>
<td>.00030</td>
<td>4.9</td>
<td>.00251</td>
<td>.00227</td>
<td>.00024</td>
<td>9.6</td>
</tr>
<tr>
<td>20-25</td>
<td>.00829</td>
<td>.00778</td>
<td>.00051</td>
<td>6.2</td>
<td>.00302</td>
<td>.00259</td>
<td>.00043</td>
<td>14.2</td>
</tr>
<tr>
<td>25-30</td>
<td>.00757</td>
<td>.00676</td>
<td>.00081</td>
<td>10.7</td>
<td>.00357</td>
<td>.00295</td>
<td>.00062</td>
<td>17.4</td>
</tr>
<tr>
<td>30-35</td>
<td>.00863</td>
<td>.00691</td>
<td>.00172</td>
<td>19.9</td>
<td>.00484</td>
<td>.00395</td>
<td>.00089</td>
<td>18.4</td>
</tr>
<tr>
<td>35-40</td>
<td>.01256</td>
<td>.00854</td>
<td>.00402</td>
<td>32.0</td>
<td>.00733</td>
<td>.00583</td>
<td>.00150</td>
<td>20.5</td>
</tr>
<tr>
<td>40-45</td>
<td>.02074</td>
<td>.01192</td>
<td>.00882</td>
<td>42.5</td>
<td>.01185</td>
<td>.00881</td>
<td>.00304</td>
<td>25.7</td>
</tr>
<tr>
<td>45-50</td>
<td>.03474</td>
<td>.01785</td>
<td>.01689</td>
<td>48.6</td>
<td>.01816</td>
<td>.01282</td>
<td>.00534</td>
<td>29.4</td>
</tr>
<tr>
<td>50-55</td>
<td>.05719</td>
<td>.02737</td>
<td>.02982</td>
<td>52.1</td>
<td>.02732</td>
<td>.01775</td>
<td>.00957</td>
<td>35.0</td>
</tr>
<tr>
<td>55-60</td>
<td>.08456</td>
<td>.03858</td>
<td>.04598</td>
<td>54.4</td>
<td>.03978</td>
<td>.02287</td>
<td>.01691</td>
<td>42.5</td>
</tr>
<tr>
<td>60-65</td>
<td>.12989</td>
<td>.05702</td>
<td>.07287</td>
<td>56.1</td>
<td>.06613</td>
<td>.03241</td>
<td>.03372</td>
<td>51.0</td>
</tr>
<tr>
<td>65-70</td>
<td>.18759</td>
<td>.07908</td>
<td>.10851</td>
<td>57.8</td>
<td>.10321</td>
<td>.04416</td>
<td>.05905</td>
<td>57.2</td>
</tr>
<tr>
<td>70-75</td>
<td>.26327</td>
<td>.10636</td>
<td>.15691</td>
<td>59.6</td>
<td>.16682</td>
<td>.06179</td>
<td>.10503</td>
<td>63.0</td>
</tr>
<tr>
<td>75-80</td>
<td>.36837</td>
<td>.14106</td>
<td>.22731</td>
<td>61.7</td>
<td>.26990</td>
<td>.08920</td>
<td>.18070</td>
<td>67.0</td>
</tr>
<tr>
<td>80-85</td>
<td>.51822</td>
<td>.19679</td>
<td>.32143</td>
<td>62.0</td>
<td>.42950</td>
<td>.13492</td>
<td>.28958</td>
<td>68.6</td>
</tr>
<tr>
<td>85-90</td>
<td>.66589</td>
<td>.25627</td>
<td>.40862</td>
<td>61.5</td>
<td>.59498</td>
<td>.19040</td>
<td>.40458</td>
<td>68.0</td>
</tr>
<tr>
<td>90-95</td>
<td>.82355</td>
<td>.35901</td>
<td>.46654</td>
<td>56.5</td>
<td>.78586</td>
<td>.29170</td>
<td>.49446</td>
<td>62.9</td>
</tr>
<tr>
<td>95+</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0</td>
<td>0</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 5
Probability of survival and the effect of eliminating CVR diseases as a cause of death, white males and females, U.S., 1960

<table>
<thead>
<tr>
<th>Age interval (in years)</th>
<th>White males</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td></td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( P_{0i} )</td>
<td>( \hat{P}_{0i+1} )</td>
<td>( \hat{P}<em>{0i+1} - P</em>{0i} )</td>
<td>( \frac{\hat{P}<em>{0i+1} - P</em>{0i}}{P_{0i}} )</td>
<td>( \hat{P}_{0i} )</td>
<td>( \hat{P}_{0i+1} )</td>
<td>( \hat{P}<em>{0i+1} - P</em>{0i} )</td>
<td>( \frac{\hat{P}<em>{0i+1} - P</em>{0i}}{P_{0i}} )</td>
<td></td>
</tr>
<tr>
<td>0 - 1</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.00000</td>
<td>0.0 %</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.00000</td>
<td>0.0 %</td>
<td></td>
</tr>
<tr>
<td>1 - 5</td>
<td>0.97385</td>
<td>0.97397</td>
<td>0.00012</td>
<td>0.0</td>
<td>0.97933</td>
<td>0.97941</td>
<td>0.00008</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>5 - 10</td>
<td>0.96977</td>
<td>0.96998</td>
<td>0.00021</td>
<td>0.0</td>
<td>0.97699</td>
<td>0.97714</td>
<td>0.00015</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>10 - 15</td>
<td>0.96716</td>
<td>0.96728</td>
<td>0.00012</td>
<td>0.0</td>
<td>0.97512</td>
<td>0.97534</td>
<td>0.00022</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>15 - 20</td>
<td>0.96467</td>
<td>0.96513</td>
<td>0.00046</td>
<td>0.0</td>
<td>0.97362</td>
<td>0.97397</td>
<td>0.00035</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>20 - 25</td>
<td>0.95871</td>
<td>0.95916</td>
<td>0.00045</td>
<td>0.1</td>
<td>0.97118</td>
<td>0.97176</td>
<td>0.00058</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>25 - 30</td>
<td>0.95076</td>
<td>0.95200</td>
<td>0.00124</td>
<td>0.2</td>
<td>0.96825</td>
<td>0.96924</td>
<td>0.00099</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>30 - 35</td>
<td>0.91356</td>
<td>0.91556</td>
<td>0.00200</td>
<td>0.2</td>
<td>0.96179</td>
<td>0.96638</td>
<td>0.00459</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>35 - 40</td>
<td>0.93542</td>
<td>0.93903</td>
<td>0.00361</td>
<td>0.4</td>
<td>0.96012</td>
<td>0.96256</td>
<td>0.00244</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>40 - 45</td>
<td>0.92367</td>
<td>0.93101</td>
<td>0.00734</td>
<td>0.8</td>
<td>0.95308</td>
<td>0.95695</td>
<td>0.00387</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>45 - 50</td>
<td>0.90451</td>
<td>0.91991</td>
<td>0.01540</td>
<td>1.7</td>
<td>0.94179</td>
<td>0.94852</td>
<td>0.00673</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>50 - 55</td>
<td>0.87309</td>
<td>0.90319</td>
<td>0.03010</td>
<td>3.5</td>
<td>0.92469</td>
<td>0.93636</td>
<td>0.01167</td>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>55 - 60</td>
<td>0.82316</td>
<td>0.87876</td>
<td>0.05560</td>
<td>6.8</td>
<td>0.89943</td>
<td>0.91794</td>
<td>0.01851</td>
<td>2.3</td>
<td></td>
</tr>
<tr>
<td>60 - 65</td>
<td>0.75355</td>
<td>0.84486</td>
<td>0.09131</td>
<td>12.1</td>
<td>0.86365</td>
<td>0.89871</td>
<td>0.03506</td>
<td>4.1</td>
<td></td>
</tr>
<tr>
<td>65 - 70</td>
<td>0.65567</td>
<td>0.79669</td>
<td>0.14102</td>
<td>21.5</td>
<td>0.80654</td>
<td>0.86958</td>
<td>0.06304</td>
<td>7.8</td>
<td></td>
</tr>
<tr>
<td>70 - 75</td>
<td>0.53267</td>
<td>0.73369</td>
<td>0.20102</td>
<td>37.7</td>
<td>0.72330</td>
<td>0.83118</td>
<td>0.10788</td>
<td>14.9</td>
<td></td>
</tr>
<tr>
<td>75 - 80</td>
<td>0.39213</td>
<td>0.65565</td>
<td>0.26322</td>
<td>67.1</td>
<td>0.60264</td>
<td>0.77982</td>
<td>0.17718</td>
<td>29.4</td>
<td></td>
</tr>
<tr>
<td>80 - 85</td>
<td>0.21787</td>
<td>0.56316</td>
<td>0.35129</td>
<td>127.2</td>
<td>0.53999</td>
<td>0.71026</td>
<td>0.27027</td>
<td>61.1</td>
<td></td>
</tr>
<tr>
<td>85 - 90</td>
<td>0.11912</td>
<td>0.45234</td>
<td>0.33292</td>
<td>278.8</td>
<td>0.25101</td>
<td>0.61443</td>
<td>0.36312</td>
<td>114.8</td>
<td></td>
</tr>
<tr>
<td>90 - 95</td>
<td>0.03990</td>
<td>0.33642</td>
<td>0.29652</td>
<td>743.2</td>
<td>0.10166</td>
<td>0.49744</td>
<td>0.39578</td>
<td>389.3</td>
<td></td>
</tr>
<tr>
<td>95+</td>
<td>0.00696</td>
<td>0.21564</td>
<td>0.20868</td>
<td>2998.3</td>
<td>0.02177</td>
<td>0.35234</td>
<td>0.33057</td>
<td>1518.5</td>
<td></td>
</tr>
</tbody>
</table>
Table 6

Expectation of life and the effect of eliminating CVR diseases as a cause of death, white males and females, U.S., 1960

<table>
<thead>
<tr>
<th>Age interval</th>
<th>White males</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>White females</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
<td>CVR present</td>
<td>CVR eliminated</td>
<td>Difference</td>
</tr>
<tr>
<td></td>
<td>( \hat{e}_i )</td>
<td>( \hat{e}_{i+1} )</td>
<td>( \hat{e}_{i+1} - \hat{e}_i )</td>
<td>( \hat{e}_i )</td>
<td>( \hat{e}_{i+1} )</td>
<td>( \hat{e}_{i+1} - \hat{e}_i )</td>
<td>( \hat{e}_i )</td>
<td>( \hat{e}_{i+1} )</td>
<td>( \hat{e}_{i+1} - \hat{e}_i )</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
<td></td>
</tr>
<tr>
<td>0 - 1</td>
<td>67.27</td>
<td>78.95</td>
<td>11.68</td>
<td>17.4%</td>
<td>74.01</td>
<td>86.77</td>
<td>12.76</td>
<td>17.2%</td>
<td></td>
</tr>
<tr>
<td>1 - 5</td>
<td>68.08</td>
<td>80.05</td>
<td>11.97</td>
<td>17.6%</td>
<td>74.50</td>
<td>87.50</td>
<td>13.00</td>
<td>17.4%</td>
<td></td>
</tr>
<tr>
<td>5 - 10</td>
<td>61.36</td>
<td>76.38</td>
<td>12.02</td>
<td>18.7%</td>
<td>70.75</td>
<td>83.79</td>
<td>13.04</td>
<td>18.4%</td>
<td></td>
</tr>
<tr>
<td>10 - 15</td>
<td>59.52</td>
<td>71.57</td>
<td>12.05</td>
<td>20.2%</td>
<td>65.88</td>
<td>78.94</td>
<td>13.06</td>
<td>19.8%</td>
<td></td>
</tr>
<tr>
<td>15 - 20</td>
<td>54.67</td>
<td>66.74</td>
<td>12.07</td>
<td>22.1%</td>
<td>60.97</td>
<td>74.05</td>
<td>13.08</td>
<td>21.5%</td>
<td></td>
</tr>
<tr>
<td>20 - 25</td>
<td>49.99</td>
<td>62.11</td>
<td>12.12</td>
<td>24.2%</td>
<td>56.12</td>
<td>69.21</td>
<td>13.09</td>
<td>23.3%</td>
<td></td>
</tr>
<tr>
<td>25 - 30</td>
<td>45.39</td>
<td>57.58</td>
<td>12.19</td>
<td>26.9%</td>
<td>51.28</td>
<td>64.38</td>
<td>13.10</td>
<td>25.5%</td>
<td></td>
</tr>
<tr>
<td>30 - 35</td>
<td>40.72</td>
<td>52.96</td>
<td>12.24</td>
<td>30.1%</td>
<td>46.16</td>
<td>59.56</td>
<td>13.10</td>
<td>28.2%</td>
<td></td>
</tr>
<tr>
<td>35 - 40</td>
<td>36.05</td>
<td>48.31</td>
<td>12.26</td>
<td>34.0%</td>
<td>41.67</td>
<td>54.79</td>
<td>13.12</td>
<td>31.5%</td>
<td></td>
</tr>
<tr>
<td>40 - 45</td>
<td>31.17</td>
<td>43.70</td>
<td>12.23</td>
<td>38.9%</td>
<td>36.96</td>
<td>50.10</td>
<td>13.14</td>
<td>35.6%</td>
<td></td>
</tr>
<tr>
<td>45 - 50</td>
<td>27.08</td>
<td>39.19</td>
<td>12.11</td>
<td>44.7%</td>
<td>32.37</td>
<td>45.52</td>
<td>13.15</td>
<td>46.3%</td>
<td></td>
</tr>
<tr>
<td>50 - 55</td>
<td>22.96</td>
<td>31.86</td>
<td>11.90</td>
<td>51.8%</td>
<td>27.92</td>
<td>41.07</td>
<td>13.15</td>
<td>47.1%</td>
<td></td>
</tr>
<tr>
<td>55 - 60</td>
<td>19.19</td>
<td>30.76</td>
<td>11.57</td>
<td>60.3%</td>
<td>23.63</td>
<td>36.77</td>
<td>13.11</td>
<td>55.6%</td>
<td></td>
</tr>
<tr>
<td>60 - 65</td>
<td>15.73</td>
<td>26.89</td>
<td>11.16</td>
<td>70.9%</td>
<td>19.50</td>
<td>32.57</td>
<td>13.07</td>
<td>67.0%</td>
<td></td>
</tr>
<tr>
<td>65 - 70</td>
<td>12.69</td>
<td>23.36</td>
<td>10.67</td>
<td>84.1%</td>
<td>15.70</td>
<td>28.57</td>
<td>12.87</td>
<td>82.0%</td>
<td></td>
</tr>
<tr>
<td>70 - 75</td>
<td>10.01</td>
<td>20.15</td>
<td>10.14</td>
<td>101.3%</td>
<td>12.20</td>
<td>24.77</td>
<td>12.57</td>
<td>103.0%</td>
<td></td>
</tr>
<tr>
<td>75 - 80</td>
<td>7.68</td>
<td>17.24</td>
<td>9.56</td>
<td>128.5%</td>
<td>9.13</td>
<td>21.23</td>
<td>12.10</td>
<td>132.5%</td>
<td></td>
</tr>
<tr>
<td>80 - 85</td>
<td>5.67</td>
<td>14.65</td>
<td>8.98</td>
<td>158.1%</td>
<td>6.57</td>
<td>18.06</td>
<td>11.49</td>
<td>174.9%</td>
<td></td>
</tr>
<tr>
<td>85 - 90</td>
<td>4.20</td>
<td>12.66</td>
<td>8.46</td>
<td>201.1%</td>
<td>4.71</td>
<td>15.51</td>
<td>10.80</td>
<td>229.3%</td>
<td></td>
</tr>
<tr>
<td>90 - 95</td>
<td>3.07</td>
<td>11.24</td>
<td>8.17</td>
<td>266.1%</td>
<td>3.32</td>
<td>13.62</td>
<td>10.30</td>
<td>310.2%</td>
<td></td>
</tr>
<tr>
<td>95+</td>
<td>2.92</td>
<td>11.39</td>
<td>8.47</td>
<td>290.1%</td>
<td>2.98</td>
<td>13.39</td>
<td>10.41</td>
<td>319.3%</td>
<td></td>
</tr>
</tbody>
</table>
Table 4 gives a comparison between $\hat{q}_i$ and $\hat{q}_{i,1}$. The difference, $\hat{q}_i - \hat{q}_{i,1}$, is the reduction in the probability of dying in age interval $(x_i, x_{i+1})$ if CVR diseases were eliminated as a risk of death, or, alternatively, the excess probability of dying due to the presence of these diseases. This difference, while not pronounced below age 30, advances with age at an accelerated rate: from .00012 (0.5%) for the first year of life to .46654 (56.5%) for age interval 90 to 95 in white males, U.S., 1960. If the effect of CVR diseases were removed, the reduction in the probability of dying for white males is 32% of the existing probability for age interval 35 to 40, over 50% for interval 50 to 55, and about 60% for interval 70 to 75. This general mortality pattern holds also for white females. The estimated probabilities $\hat{q}_i$ and $\hat{q}_{i,1}$, and their difference $\hat{q}_i - \hat{q}_{i,1}$ are lower for females than for males up to age 90. The relative reduction in the probability of dying is for females about 20% for age interval 35 to 40, 35% for interval 50 to 55, and 63% for interval 70 to 75. In fact from age 30 to age 70, the relative reduction in the probability of dying is lower for females than for males, but the reverse is true for other ages. Thus, relatively speaking, from age 70 on cardiovascular-renal diseases have a larger impact on white females than white males, although in absolute terms these diseases contribute more deaths in the white male population than in the white female population almost throughout life.

The impact of CVR diseases on the probability of survival is shown in Table 5, where $P_{0i} = \frac{l_i}{l_0}$ is taken from the life table of the entire white population for each sex, and $P_{0i,1} = \frac{l_{i,1}}{l_{0,1}}$ is from Tables 2 and 3. Although the impact on the probability of survival is less pronounced than the probability of dying in the younger ages, it is much more alarming in the older age groups. From age 30 on for males, and from age 40 on for
females, the relative reduction in the survival probability due to the presence of CVR has been doubled over every 5 year age interval.

Table 6 shows that these diseases cause an average loss of 12 years in the expectation of life for white males under age 50 and 13 years for females under age 65. At older ages, the loss in the expectation of life decreases slightly in absolute value but increases spectacularly relative to the existing life expectancy. If CVR diseases were eliminated as a risk of death, a male could expect an 30% increase of length of life over the present life expectancy at 30 years of age, 50% at age 50 and 100% at age 70. Comparable percentages of increased lengths of life that could be expected at these ages (28% at age 30, 47% at age 50 and 103% at age 70) are found for a female.

4.1 Comparison of impact on human mortality of three major causes of deaths: All accidents, cancer all forms, and cardiovascular-renal diseases

Different diseases have definite effects on human mortality and longevity. Concerted efforts are being made through the World Health Organization and health programs of individual countries to reduce mortality due to specific diseases. Relative importance of diseases as causes of death play a significant role in determining the priority in overall health planning. The purpose of this section is to show how some leading causes of death may be compared using the life table and competing risk methodology.

Tables 7, 8 and 9, are the life tables of the Federal Republic of Germany 1970 population when cardiovascular-renal diseases \( R_1 \), cancer all forms \( R_2 \), and all accidents \( R_3 \) respectively, are eliminated as causes of death. Each table shows a hypothetical pattern that would exist in the Federal Republic of Germany if the corresponding diseases were eliminated.
Table 7.

Life Table of the Federal Republic of Germany population, 1970 when cardiovascular diseases ($R_1$) are eliminated as a cause of death.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval $(x_i, x_{i+1})$</th>
<th>Number living at age $x_i$</th>
<th>Number dying in interval $(x_i, x_{i+1})$</th>
<th>Fraction of last age interval of life lived in interval beyond age $x_i$</th>
<th>Number of years lived beyond age $x_i$</th>
<th>Total number of years lived at age $x_i$</th>
<th>Observed Expectation of life at age $x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x_i to x_{i+1}</td>
<td>$\hat{q}_{i.1}$</td>
<td>$l_{i.1}$</td>
<td>$d_{i.1}$</td>
<td>$a_{i}$</td>
<td>$L_{i.1}$</td>
<td>$T_{i.1}$</td>
<td>$\hat{e}_{i.1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>0.02117</td>
<td>100000</td>
<td>2117</td>
<td>0.10</td>
<td>98095</td>
<td>7749467</td>
<td>77.49</td>
</tr>
<tr>
<td>1-5</td>
<td>0.00374</td>
<td>97883</td>
<td>366</td>
<td>0.39</td>
<td>390639</td>
<td>7651372</td>
<td>78.17</td>
</tr>
<tr>
<td>5-10</td>
<td>0.00257</td>
<td>97517</td>
<td>251</td>
<td>0.46</td>
<td>486907</td>
<td>7260733</td>
<td>74.46</td>
</tr>
<tr>
<td>10-15</td>
<td>0.00201</td>
<td>97266</td>
<td>196</td>
<td>0.52</td>
<td>485860</td>
<td>6773826</td>
<td>69.64</td>
</tr>
<tr>
<td>15-20</td>
<td>0.00505</td>
<td>97070</td>
<td>490</td>
<td>0.57</td>
<td>484296</td>
<td>6287966</td>
<td>64.78</td>
</tr>
<tr>
<td>20-25</td>
<td>0.00573</td>
<td>96580</td>
<td>553</td>
<td>0.52</td>
<td>481573</td>
<td>5803670</td>
<td>60.09</td>
</tr>
<tr>
<td>25-30</td>
<td>0.00511</td>
<td>96027</td>
<td>491</td>
<td>0.51</td>
<td>478932</td>
<td>5322097</td>
<td>55.42</td>
</tr>
<tr>
<td>30-35</td>
<td>0.00633</td>
<td>95536</td>
<td>605</td>
<td>0.52</td>
<td>476228</td>
<td>4843165</td>
<td>50.69</td>
</tr>
<tr>
<td>35-40</td>
<td>0.00832</td>
<td>94931</td>
<td>790</td>
<td>0.54</td>
<td>472838</td>
<td>4366937</td>
<td>46.00</td>
</tr>
<tr>
<td>40-45</td>
<td>0.01138</td>
<td>94141</td>
<td>1071</td>
<td>0.53</td>
<td>468188</td>
<td>3894099</td>
<td>41.36</td>
</tr>
<tr>
<td>45-50</td>
<td>0.01630</td>
<td>93070</td>
<td>1517</td>
<td>0.51</td>
<td>461633</td>
<td>3425911</td>
<td>36.81</td>
</tr>
<tr>
<td>50-55</td>
<td>0.02481</td>
<td>91553</td>
<td>2271</td>
<td>0.58</td>
<td>452996</td>
<td>2964278</td>
<td>32.38</td>
</tr>
<tr>
<td>55-60</td>
<td>0.03529</td>
<td>89282</td>
<td>3151</td>
<td>0.54</td>
<td>439163</td>
<td>2511282</td>
<td>28.13</td>
</tr>
<tr>
<td>60-65</td>
<td>0.05576</td>
<td>86131</td>
<td>4803</td>
<td>0.54</td>
<td>419608</td>
<td>2072119</td>
<td>24.06</td>
</tr>
<tr>
<td>65-70</td>
<td>0.08802</td>
<td>81328</td>
<td>7158</td>
<td>0.52</td>
<td>389461</td>
<td>1652511</td>
<td>20.32</td>
</tr>
<tr>
<td>70-75</td>
<td>0.12736</td>
<td>74170</td>
<td>9446</td>
<td>0.52</td>
<td>348180</td>
<td>1263050</td>
<td>17.03</td>
</tr>
<tr>
<td>75-80</td>
<td>0.18092</td>
<td>64724</td>
<td>11710</td>
<td>0.51</td>
<td>294930</td>
<td>914870</td>
<td>14.13</td>
</tr>
<tr>
<td>80-85</td>
<td>0.25626</td>
<td>53014</td>
<td>13585</td>
<td>0.49</td>
<td>230428</td>
<td>659940</td>
<td>11.69</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>39429</td>
<td>39429</td>
<td>0.49</td>
<td>389512</td>
<td>389512</td>
<td>9.88</td>
</tr>
</tbody>
</table>
Table 8.

Life Table of the Federal Republic of Germany population, 1970 when cancer all forms \( R_2 \) is eliminated as a cause of death.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval ( x_i ) to ( x_{i+1} )</th>
<th>Number living at age ( x_i )</th>
<th>Number dying in interval ( x_i ) to ( x_{i+1} )</th>
<th>Fraction of last age interval of life</th>
<th>Number of years lived in interval beyond age ( x_i )</th>
<th>Total number of years lived</th>
<th>Observed Expectation of life at age ( x_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.02117</td>
<td>100000</td>
<td>2117</td>
<td>.10</td>
<td>98095</td>
<td>7323376</td>
<td>73.23</td>
</tr>
<tr>
<td>1-5</td>
<td>.00343</td>
<td>97883</td>
<td>336</td>
<td>.39</td>
<td>390712</td>
<td>7225281</td>
<td>73.82</td>
</tr>
<tr>
<td>5-10</td>
<td>.00227</td>
<td>97547</td>
<td>221</td>
<td>.46</td>
<td>487138</td>
<td>6834569</td>
<td>70.06</td>
</tr>
<tr>
<td>10-15</td>
<td>.00182</td>
<td>97326</td>
<td>177</td>
<td>.52</td>
<td>486205</td>
<td>6347431</td>
<td>65.22</td>
</tr>
<tr>
<td>15-20</td>
<td>.00481</td>
<td>97149</td>
<td>467</td>
<td>.57</td>
<td>484741</td>
<td>5861226</td>
<td>60.33</td>
</tr>
<tr>
<td>20-25</td>
<td>.00553</td>
<td>96682</td>
<td>535</td>
<td>.52</td>
<td>482126</td>
<td>5376485</td>
<td>55.61</td>
</tr>
<tr>
<td>25-30</td>
<td>.00485</td>
<td>96147</td>
<td>466</td>
<td>.51</td>
<td>479593</td>
<td>4894359</td>
<td>50.90</td>
</tr>
<tr>
<td>30-35</td>
<td>.00603</td>
<td>95681</td>
<td>577</td>
<td>.52</td>
<td>477020</td>
<td>4414766</td>
<td>46.14</td>
</tr>
<tr>
<td>35-40</td>
<td>.00808</td>
<td>95104</td>
<td>768</td>
<td>.54</td>
<td>473754</td>
<td>3937746</td>
<td>41.40</td>
</tr>
<tr>
<td>40-45</td>
<td>.01116</td>
<td>94336</td>
<td>1053</td>
<td>.53</td>
<td>469205</td>
<td>3463992</td>
<td>36.72</td>
</tr>
<tr>
<td>45-50</td>
<td>.01595</td>
<td>93283</td>
<td>1488</td>
<td>.51</td>
<td>462769</td>
<td>2994787</td>
<td>32.10</td>
</tr>
<tr>
<td>50-55</td>
<td>.02433</td>
<td>91795</td>
<td>2233</td>
<td>.58</td>
<td>454286</td>
<td>2532018</td>
<td>27.58</td>
</tr>
<tr>
<td>55-60</td>
<td>.03681</td>
<td>89562</td>
<td>3297</td>
<td>.54</td>
<td>440227</td>
<td>2077732</td>
<td>23.20</td>
</tr>
<tr>
<td>60-65</td>
<td>.06501</td>
<td>86265</td>
<td>5608</td>
<td>.54</td>
<td>418427</td>
<td>1637505</td>
<td>18.98</td>
</tr>
<tr>
<td>65-70</td>
<td>.11328</td>
<td>80657</td>
<td>9137</td>
<td>.52</td>
<td>381356</td>
<td>1219078</td>
<td>15.11</td>
</tr>
<tr>
<td>70-75</td>
<td>.18512</td>
<td>71520</td>
<td>13240</td>
<td>.52</td>
<td>325824</td>
<td>837722</td>
<td>11.71</td>
</tr>
<tr>
<td>75-80</td>
<td>.29308</td>
<td>58280</td>
<td>17081</td>
<td>.51</td>
<td>249552</td>
<td>511898</td>
<td>8.78</td>
</tr>
<tr>
<td>80-85</td>
<td>.44942</td>
<td>41199</td>
<td>18516</td>
<td>.49</td>
<td>158779</td>
<td>262346</td>
<td>6.37</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>22683</td>
<td>22683</td>
<td></td>
<td></td>
<td></td>
<td>4.57</td>
</tr>
</tbody>
</table>
Life Table of the Federal Republic of Germany population, 1970 when all accidents ($R_3$) are eliminated as a cause of death.

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval ($x_i, x_{i+1}$)</th>
<th>Number living at age $x_i$</th>
<th>Number dying in interval ($x_i, x_{i+1}$)</th>
<th>Fraction of last age of interval of lived</th>
<th>Number of years lived beyond age $x_i$</th>
<th>Total number of years lived</th>
<th>Observed Expectation of life at age $x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$ to $x_{i+1}$</td>
<td>$q_{i.3}$</td>
<td>$d_{i.3}$</td>
<td>$d_{i.3}$</td>
<td>$a_i$</td>
<td>$L_{i.3}$</td>
<td>$T_{i.3}$</td>
<td>$e_{i.3}$</td>
</tr>
<tr>
<td>-------------------------</td>
<td>---------------------------------</td>
<td>-----------------</td>
<td>---------------------------------</td>
<td>-----------------------------------</td>
<td>-----------------------------</td>
<td>-----------------------------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>0-1</td>
<td>.02054</td>
<td>100000</td>
<td>2754</td>
<td>.10</td>
<td>98151</td>
<td>7195696</td>
<td>71.96</td>
</tr>
<tr>
<td>1-5</td>
<td>.00253</td>
<td>97946</td>
<td>248</td>
<td>.39</td>
<td>391179</td>
<td>7097545</td>
<td>72.46</td>
</tr>
<tr>
<td>5-10</td>
<td>.00123</td>
<td>97698</td>
<td>120</td>
<td>.46</td>
<td>488166</td>
<td>6706366</td>
<td>68.64</td>
</tr>
<tr>
<td>10-15</td>
<td>.00112</td>
<td>97578</td>
<td>109</td>
<td>.52</td>
<td>487628</td>
<td>6218200</td>
<td>63.73</td>
</tr>
<tr>
<td>15-20</td>
<td>.00203</td>
<td>97469</td>
<td>198</td>
<td>.57</td>
<td>486919</td>
<td>5730572</td>
<td>58.79</td>
</tr>
<tr>
<td>20-25</td>
<td>.00271</td>
<td>97271</td>
<td>264</td>
<td>.52</td>
<td>485721</td>
<td>5243653</td>
<td>53.91</td>
</tr>
<tr>
<td>25-30</td>
<td>.00334</td>
<td>97007</td>
<td>324</td>
<td>.51</td>
<td>484241</td>
<td>4757932</td>
<td>49.05</td>
</tr>
<tr>
<td>30-35</td>
<td>.00305</td>
<td>96683</td>
<td>488</td>
<td>.52</td>
<td>482244</td>
<td>4273691</td>
<td>44.20</td>
</tr>
<tr>
<td>35-40</td>
<td>.00790</td>
<td>96195</td>
<td>760</td>
<td>.54</td>
<td>479227</td>
<td>3791447</td>
<td>39.41</td>
</tr>
<tr>
<td>40-45</td>
<td>.01287</td>
<td>95435</td>
<td>1228</td>
<td>.53</td>
<td>474289</td>
<td>3312220</td>
<td>34.71</td>
</tr>
<tr>
<td>45-50</td>
<td>.02048</td>
<td>94207</td>
<td>1929</td>
<td>.51</td>
<td>466309</td>
<td>2837931</td>
<td>30.12</td>
</tr>
<tr>
<td>50-55</td>
<td>.03297</td>
<td>92278</td>
<td>3042</td>
<td>.58</td>
<td>455002</td>
<td>2371622</td>
<td>25.70</td>
</tr>
<tr>
<td>55-60</td>
<td>.05051</td>
<td>89236</td>
<td>4507</td>
<td>.54</td>
<td>435814</td>
<td>1916620</td>
<td>21.48</td>
</tr>
<tr>
<td>60-65</td>
<td>.08632</td>
<td>84729</td>
<td>7314</td>
<td>.54</td>
<td>406823</td>
<td>1480806</td>
<td>17.48</td>
</tr>
<tr>
<td>65-70</td>
<td>.14551</td>
<td>77415</td>
<td>11265</td>
<td>.52</td>
<td>360039</td>
<td>1073983</td>
<td>13.87</td>
</tr>
<tr>
<td>70-75</td>
<td>.22526</td>
<td>66150</td>
<td>14901</td>
<td>.52</td>
<td>294988</td>
<td>713944</td>
<td>10.79</td>
</tr>
<tr>
<td>75-80</td>
<td>.33585</td>
<td>51249</td>
<td>17212</td>
<td>.51</td>
<td>214076</td>
<td>418956</td>
<td>8.17</td>
</tr>
<tr>
<td>80-85</td>
<td>.48554</td>
<td>34037</td>
<td>16526</td>
<td>.49</td>
<td>128044</td>
<td>204880</td>
<td>6.02</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>17511</td>
<td>17511</td>
<td>.49</td>
<td>76836</td>
<td>76836</td>
<td>4.39</td>
</tr>
</tbody>
</table>
For comparison, Tables 10a and 10b are reproduced from the life tables to show the differences \( q_{i} - q_{i,1} \), \( q_{i} - q_{i,2} \) and \( q_{i} - q_{i,3} \), for each age group. These differences represent the increase in probability of dying due to the presence of the corresponding disease. Table 10b shows that the contribution of accidents to the probability of dying is quite uniform over most of the life span with the exception of very old ages where some increase has taken place. On the other hand, the differences in probability of dying for cancer all forms and cardiovascular-renal diseases are not significant for ages less than 30 years, but they increase rapidly with the advancement of age. The differences are higher for cancer than cardiovascular-renal diseases for age groups below 55, but the reverse is true for older ages. For age group 80 to 85, the difference for cardiovascular-renal diseases is more than four times as large as that for all cancers. Since the probability of dying in old age groups is much higher than in younger age groups, cardiovascular-renal diseases have a greater effect on human longevity than does cancer.

The relative impact of these three causes of death on human longevity becomes quite clear in Table 11. In this Table, the expectation of life \( \hat{e}_{1} \) at each age when all risks are operating is being compared with the expectation of life \( \hat{e}_{i,1} \), \( \hat{e}_{i,2} \), and \( \hat{e}_{i,3} \) when one of the causes is eliminated. We see that cardiovascular diseases are by far the most important causes of death. Cancer all forms runs a distant second, and all accidents a poor third. The difference \( \hat{e}_{i,1} - \hat{e}_{i} \) is quite constant throughout the life span. The figures show that the presence of cardiovascular-renal diseases has, on the average, cost the people in the Federal Republic of Germany about 7 years loss of life. The corresponding difference for cancer decreases as age increases, with the largest difference of 2.58 years at age 1-5 and the smallest of .39 years.
Table 10a.

Probability of dying when cardiovascular diseases (R₁), cancer all forms (R₂), or all accidents (R₃) are eliminated as a cause of death.

(The Federal Republic of Germany, 1970)

<table>
<thead>
<tr>
<th>Age Interval (in years)</th>
<th>Probability of dying in interval (xᵢ, xᵢ₊₁)</th>
<th>Probability of Dying When A Cause is Eliminated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>qᵢ</td>
<td>qᵢ₈</td>
</tr>
<tr>
<td>0-1</td>
<td>0.02123</td>
<td>0.02117</td>
</tr>
<tr>
<td>1-5</td>
<td>0.00379</td>
<td>0.00374</td>
</tr>
<tr>
<td>5-10</td>
<td>0.00260</td>
<td>0.00257</td>
</tr>
<tr>
<td>10-15</td>
<td>0.00208</td>
<td>0.00201</td>
</tr>
<tr>
<td>15-20</td>
<td>0.00516</td>
<td>0.00505</td>
</tr>
<tr>
<td>20-25</td>
<td>0.00598</td>
<td>0.00573</td>
</tr>
<tr>
<td>25-30</td>
<td>0.00548</td>
<td>0.00511</td>
</tr>
<tr>
<td>30-35</td>
<td>0.00707</td>
<td>0.00633</td>
</tr>
<tr>
<td>35-40</td>
<td>0.00991</td>
<td>0.00832</td>
</tr>
<tr>
<td>40-45</td>
<td>0.01471</td>
<td>0.01138</td>
</tr>
<tr>
<td>45-50</td>
<td>0.02224</td>
<td>0.01636</td>
</tr>
<tr>
<td>50-55</td>
<td>0.03499</td>
<td>0.02481</td>
</tr>
<tr>
<td>55-60</td>
<td>0.05275</td>
<td>0.03529</td>
</tr>
<tr>
<td>60-65</td>
<td>0.08915</td>
<td>0.05576</td>
</tr>
<tr>
<td>65-70</td>
<td>0.14877</td>
<td>0.08802</td>
</tr>
<tr>
<td>70-75</td>
<td>0.23005</td>
<td>0.12736</td>
</tr>
<tr>
<td>75-80</td>
<td>0.34382</td>
<td>0.18092</td>
</tr>
<tr>
<td>80-85</td>
<td>0.49841</td>
<td>0.25626</td>
</tr>
<tr>
<td>85+</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>Interval</td>
<td>Cardiovascular diseases, $R_1$</td>
<td>Cancer all forms, $R_2$</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>$x_1$ to $x_{1+1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\hat{q}<em>1 - \hat{q}</em>{1.1}$</td>
<td>$\hat{q}<em>1 - \hat{q}</em>{1.2}$</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>0 - 1</td>
<td>.00006</td>
<td>.00006</td>
</tr>
<tr>
<td>1 - 5</td>
<td>.00005</td>
<td>.00036</td>
</tr>
<tr>
<td>5 - 10</td>
<td>.00003</td>
<td>.00033</td>
</tr>
<tr>
<td>10 - 15</td>
<td>.00005</td>
<td>.00024</td>
</tr>
<tr>
<td>15 - 20</td>
<td>.00011</td>
<td>.00035</td>
</tr>
<tr>
<td>20 - 25</td>
<td>.00025</td>
<td>.00045</td>
</tr>
<tr>
<td>25 - 30</td>
<td>.00037</td>
<td>.00063</td>
</tr>
<tr>
<td>30 - 35</td>
<td>.00074</td>
<td>.00104</td>
</tr>
<tr>
<td>35 - 40</td>
<td>.00159</td>
<td>.00183</td>
</tr>
<tr>
<td>40 - 45</td>
<td>.00333</td>
<td>.00355</td>
</tr>
<tr>
<td>45 - 50</td>
<td>.00594</td>
<td>.00629</td>
</tr>
<tr>
<td>50 - 55</td>
<td>.01018</td>
<td>.01066</td>
</tr>
<tr>
<td>55 - 60</td>
<td>.01746</td>
<td>.01594</td>
</tr>
<tr>
<td>60 - 65</td>
<td>.03339</td>
<td>.02414</td>
</tr>
<tr>
<td>65 - 70</td>
<td>.06075</td>
<td>.03549</td>
</tr>
<tr>
<td>70 - 75</td>
<td>.10269</td>
<td>.04493</td>
</tr>
<tr>
<td>75 - 80</td>
<td>.16290</td>
<td>.05074</td>
</tr>
<tr>
<td>80 - 85</td>
<td>.24215</td>
<td>.04899</td>
</tr>
<tr>
<td>85 +</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE 10b
Probability of dying and the effect of eliminating cardiovascular diseases ($R_1$), cancer all forms ($R_2$), or all accidents ($R_3$) as a cause of death in each age interval.

Table 11.

Expectation of life and the effect of elimination of cardiovascular diseases \((R_1)\), Cancer all forms \((R_2)\), or all accidents \((R_3)\) as a cause of death in each age interval.


<table>
<thead>
<tr>
<th>Age Interval ((x_i \text{ to } x_{i+1}))</th>
<th>Observed Expectation of Life (\hat{e}_i)</th>
<th>Expectation of life with elimination as cause of death</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Cardiovascular Diseases 1.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\hat{e}_i.1)</td>
</tr>
<tr>
<td>0-1</td>
<td>70.71</td>
<td>77.49</td>
</tr>
<tr>
<td>1-5</td>
<td>71.24</td>
<td>78.17</td>
</tr>
<tr>
<td>5-10</td>
<td>67.51</td>
<td>74.46</td>
</tr>
<tr>
<td>10-15</td>
<td>62.68</td>
<td>69.64</td>
</tr>
<tr>
<td>15-20</td>
<td>57.80</td>
<td>64.78</td>
</tr>
<tr>
<td>20-25</td>
<td>53.09</td>
<td>60.09</td>
</tr>
<tr>
<td>25-30</td>
<td>48.39</td>
<td>55.42</td>
</tr>
<tr>
<td>30-35</td>
<td>43.64</td>
<td>50.69</td>
</tr>
<tr>
<td>35-40</td>
<td>38.94</td>
<td>46.00</td>
</tr>
<tr>
<td>40-45</td>
<td>34.30</td>
<td>41.36</td>
</tr>
<tr>
<td>45-50</td>
<td>29.77</td>
<td>36.81</td>
</tr>
<tr>
<td>50-55</td>
<td>25.39</td>
<td>32.38</td>
</tr>
<tr>
<td>55-60</td>
<td>21.21</td>
<td>28.13</td>
</tr>
<tr>
<td>60-65</td>
<td>17.24</td>
<td>24.06</td>
</tr>
<tr>
<td>65-70</td>
<td>13.66</td>
<td>20.32</td>
</tr>
<tr>
<td>70-75</td>
<td>10.59</td>
<td>17.03</td>
</tr>
<tr>
<td>75-80</td>
<td>7.98</td>
<td>14.13</td>
</tr>
<tr>
<td>80-85</td>
<td>5.82</td>
<td>11.69</td>
</tr>
<tr>
<td>85+</td>
<td>4.18</td>
<td>9.88</td>
</tr>
</tbody>
</table>

1. Cardiovascular diseases \((A80-A88)\)
2. Cancer all forms \((A45-A60)\)
3. All accidents \((AE138-AE146)\)
Table 12.
Probability of dying and the effect of eliminating cancer all forms \( (R_2) \)
as a cause of death

(Canada 1968 and France 1969)

| Age Interval (in years) \((x_i, x_{i+1})\) | Male \(\hat{q}_{i\mid i+1} - \hat{q}_{i+1|2}\) | Female \(\hat{q}_{i\mid i+1} - \hat{q}_{i+1|2}\) |
|---|---|---|
| Canada | France | Canada | France |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| 1-5 | .00019 | .00022 | 4.7% | 6.8% | .00029 | 7.8% | .00025 | 8.2% |
| 5-10 | .00033 | .00019 | 11.1% | 9.6% | .00037 | 16.2% | .00034 | 20.5% |
| 10-15 | .00015 | .00013 | 5.8% | 8.4% | .00062 | 25.8% | .00033 | 22.9% |
| 15-20 | .00048 | .00034 | 7.5% | 13.6% | .00126 | 20.8% | .00058 | 21.3% |
| 20-25 | .00030 | .00042 | 3.3% | 14.9% | .00097 | 12.1% | .00017 | 5.2% |
| 25-30 | .00077 | .00058 | 10.3% | 18.2% | .00123 | 15.3% | .00050 | 13.7% |
| 30-35 | .00106 | .00058 | 13.2% | 13.6% | .00141 | 14.4% | .00057 | 12.4% |
| 35-40 | .00171 | .00244 | 15.5% | 37.6% | .00313 | 21.3% | .00182 | 25.1% |
| 40-45 | .00206 | .00422 | 12.0% | 41.6% | .00520 | 23.2% | .00378 | 34.2% |
| 45-50 | .00506 | .00793 | 17.8% | 44.0% | .00867 | 25.3% | .00662 | 39.1% |
| 50-55 | .00849 | .01037 | 18.6% | 42.1% | .01534 | 29.0% | .00913 | 36.5% |
| 55-60 | .01614 | .01432 | 22.0% | 37.8% | .02429 | 29.8% | .01405 | 30.1% |
| 60-65 | .02326 | .01801 | 20.8% | 31.3% | .03925 | 31.2% | .01922 | 34.9% |
| 65-70 | .03685 | .02348 | 22.2% | 26.0% | .05564 | 29.9% | .02799 | 31.3% |
| 70-75 | .04801 | .03010 | 20.7% | 21.4% | .07125 | 26.9% | .04070 | 26.6% |
| 75-80 | .04930 | .03325 | 14.8% | 14.7% | .08024 | 21.4% | .05281 | 20.8% |
| 80-85 | .06147 | .03818 | 13.1% | 10.3% | .07075 | 13.5% | .05234 | 13.0% |
Table 13.
Expectation of life and the effect of eliminating cancer all forms \( (R_2) \) as a cause of death

(Canada 1968 and France 1969)

<table>
<thead>
<tr>
<th>Age Interval ((x_i, x_{i+1}))</th>
<th>Canada</th>
<th></th>
<th>France</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\hat{e}_1)</td>
<td>(\hat{e}_{1.2}-\hat{e}_1)</td>
<td>(\hat{e}_1)</td>
<td>(\hat{e}_{1.2}-\hat{e}_1)</td>
<td>(\hat{e}_1)</td>
<td>(\hat{e}_{1.2}-\hat{e}_1)</td>
<td>(\hat{e}_1)</td>
<td>(\hat{e}_{1.2}-\hat{e}_1)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
</tr>
<tr>
<td>0-1</td>
<td>69.04</td>
<td>2.37</td>
<td>75.69</td>
<td>2.75</td>
<td>67.82</td>
<td>3.58</td>
<td>75.38</td>
<td>3.12</td>
</tr>
<tr>
<td>1-5</td>
<td>69.66</td>
<td>2.47</td>
<td>76.09</td>
<td>2.87</td>
<td>68.11</td>
<td>3.66</td>
<td>75.50</td>
<td>3.15</td>
</tr>
<tr>
<td>5-10</td>
<td>65.93</td>
<td>2.47</td>
<td>72.33</td>
<td>2.87</td>
<td>64.36</td>
<td>3.65</td>
<td>71.73</td>
<td>3.14</td>
</tr>
<tr>
<td>10-15</td>
<td>61.12</td>
<td>2.45</td>
<td>67.47</td>
<td>2.86</td>
<td>59.50</td>
<td>3.64</td>
<td>66.84</td>
<td>3.12</td>
</tr>
<tr>
<td>15-20</td>
<td>56.27</td>
<td>2.45</td>
<td>62.57</td>
<td>2.85</td>
<td>54.64</td>
<td>3.61</td>
<td>61.93</td>
<td>3.11</td>
</tr>
<tr>
<td>20-25</td>
<td>51.61</td>
<td>2.44</td>
<td>57.72</td>
<td>2.84</td>
<td>49.96</td>
<td>3.56</td>
<td>57.09</td>
<td>3.08</td>
</tr>
<tr>
<td>25-30</td>
<td>47.07</td>
<td>2.45</td>
<td>52.88</td>
<td>2.82</td>
<td>45.34</td>
<td>3.54</td>
<td>52.27</td>
<td>3.08</td>
</tr>
<tr>
<td>30-35</td>
<td>42.41</td>
<td>2.42</td>
<td>48.04</td>
<td>2.80</td>
<td>40.69</td>
<td>3.51</td>
<td>47.45</td>
<td>3.07</td>
</tr>
<tr>
<td>35-40</td>
<td>37.73</td>
<td>2.40</td>
<td>43.23</td>
<td>2.77</td>
<td>36.06</td>
<td>3.49</td>
<td>42.66</td>
<td>3.05</td>
</tr>
<tr>
<td>40-45</td>
<td>33.12</td>
<td>2.36</td>
<td>38.50</td>
<td>2.68</td>
<td>31.56</td>
<td>3.42</td>
<td>37.95</td>
<td>3.00</td>
</tr>
<tr>
<td>45-50</td>
<td>28.65</td>
<td>2.33</td>
<td>33.86</td>
<td>2.54</td>
<td>27.23</td>
<td>3.32</td>
<td>33.35</td>
<td>2.88</td>
</tr>
<tr>
<td>50-55</td>
<td>24.41</td>
<td>2.25</td>
<td>29.37</td>
<td>2.34</td>
<td>23.10</td>
<td>3.18</td>
<td>28.88</td>
<td>2.69</td>
</tr>
<tr>
<td>55-60</td>
<td>20.45</td>
<td>2.14</td>
<td>25.05</td>
<td>2.08</td>
<td>19.24</td>
<td>2.96</td>
<td>24.55</td>
<td>2.49</td>
</tr>
<tr>
<td>60-65</td>
<td>16.86</td>
<td>1.93</td>
<td>20.92</td>
<td>1.80</td>
<td>15.71</td>
<td>2.68</td>
<td>20.37</td>
<td>2.22</td>
</tr>
<tr>
<td>65-70</td>
<td>13.65</td>
<td>1.71</td>
<td>17.04</td>
<td>1.50</td>
<td>12.60</td>
<td>2.28</td>
<td>16.40</td>
<td>1.93</td>
</tr>
<tr>
<td>70-75</td>
<td>10.85</td>
<td>1.41</td>
<td>13.46</td>
<td>1.22</td>
<td>9.88</td>
<td>1.83</td>
<td>12.74</td>
<td>1.61</td>
</tr>
<tr>
<td>75-80</td>
<td>8.36</td>
<td>1.07</td>
<td>10.24</td>
<td>.94</td>
<td>7.53</td>
<td>1.37</td>
<td>9.56</td>
<td>1.28</td>
</tr>
<tr>
<td>80-85</td>
<td>6.25</td>
<td>.88</td>
<td>7.46</td>
<td>.78</td>
<td>5.54</td>
<td>1.01</td>
<td>6.93</td>
<td>.99</td>
</tr>
<tr>
<td>85+</td>
<td>4.61</td>
<td>.73</td>
<td>5.37</td>
<td>.76</td>
<td>4.01</td>
<td>.91</td>
<td>4.90</td>
<td>.93</td>
</tr>
</tbody>
</table>
at age 85. The average loss of length of life due to cancer is about 2.0 years. The length of life lost due to all accidents also decreases with the advancement of age. At age 0, the loss is 1.25, while at age 75, .19 years. On the average the loss due to all accidents is less than one year.

It may be noted that, in comparison with the findings in Table 6, cardiovascular-renal diseases are a more serious cause of death in the United States than they are in the Federal Republic of Germany.

4.2. Cancer all forms

Cancer all forms is next only to heart disease as a major cause of death. It claimed about 17 percent of all deaths in the United States in recent years. In spite of immeasurable amounts of scientific research effort, the cause of the disease is still unknown, and effective treatment is yet to be found. Concern has been expressed regarding the susceptibility to the disease as a function of age, sex, race, socio-economic status and others. To show how these diseases affect longevity of people of different ages, sex, and locality, we have computed the probability of dying \( \hat{q}_{1.2} \) when cancer all forms is eliminated as a risk of death and the corresponding expectation of life \( \hat{e}_{1.2} \) for the populations of Canada and France. The findings are recorded in Tables 12 and 13. For both the Canadian and French males (age 25-80) the difference \( \hat{q}_{1} - \hat{q}_{1.2} \) increases as age advances, although the effect on French males is more pronounced. The reverse pattern holds for females between ages 20 and 60, where the Canadian females are more affected than the French females. In the age interval from 35 to 55 in Canada, the difference between the two probabilities is greater for females than for males. This may be attributable to the prevalence of breast cancer among women.
The number of years of life lost due to cancer all forms is greater for the French population than for the Canadian population for both sexes and all age categories. In France, the males would gain more years of life than females if cancer all forms was eliminated as a risk of death, while in Canada females would gain more years of life than males up to age 55.

5. The Life Table when a Particular Cause Alone is Operating in a Population

The procedure in constructing a life table when a particular risk is the only risk operating in a population is also the same as that described in Section 2 of this Chapter, except for the difference in the basic quantities. As an example, let us consider the net probability of dying, \( q_{i1} \), when risk \( R_1 \) is the only risk acting. Since \( q_{i1} \) cannot be estimated directly, we make use of the result in the competing risks and estimate \( q_{i1} \) from the formula (cf., Equation (2.21a) in Appendix III),

\[
q_{i1} = Q_{i1} \left( 1 + \frac{1}{2} (q_i - Q_{i1}) \right) \quad (5.1)
\]

When a life table is for a current population, \( q_i \) and \( Q_{i1} \) are estimated, as in Section 2, from

\[
\hat{q}_i = \frac{n_i M_i}{1 + (1-a_i) n_i M_i} \quad (2.4)
\]

and

\[
\hat{Q}_{i1} = \frac{n_i M_i}{1 + (1-a_i) n_i M_i} \quad (2.5)
\]

and hence,

\[
\hat{q}_{i1} = \hat{Q}_{i1} \left[ 1 + \frac{1}{2} (\hat{q}_i - \hat{Q}_{i1}) \right] \quad (5.2)
\]
For the last age interval, e.g., 85 and over, $q_{85,1} = 1$. When all the $q_{i1}$ have been computed, we assume a radix $l_0 = 100,000$ and proceed to construct the rest of the table in the same way as before. We shall not repeat the description.
CHAPTER 9

MEDICAL FOLLOW-UP STUDIES

1. Introduction

Statistical studies in the general category of medical follow-up and life testing have as their common immediate objective the estimation of life expectancy and survival rates for a defined population at risk. Such studies usually must be terminated before all survival information is complete and are therefore said to be truncated. The nature of the problem in an investigation concerned with the medical follow-up of patients is the same as in the life testing of electric bulbs, although differences in sample size may require different approaches. For illustration, we use cancer survival data of a large sample and therefore our terminology is the same as that of the medical follow-up study.

In a typical follow-up study, a group of individuals with some common morbidity experience is followed from a well-defined zero point, such as date of hospital admission. The purpose of the study might be to evaluate a certain therapeutic measure by comparing the expectation of life and survival rates of treated patients with those of untreated patients, or by comparing the expectation of life of treated and presumably cured patients with that of the general population. When the period of observation ends, there will usually remain a number of individuals for whom the mortality data is incomplete. First, some patients will still be alive at the close of the study.
Second, some patients will have died from causes other than those under study, so that the chance of dying from the specific cause cannot be determined directly. Finally, patients will be "lost" to the study because of follow-up failure. These three sources of incomplete information have created interesting statistical problems in the estimation of the expectation of life and survival rates. Many contributions have been made to methods of analysis of follow-up data. They include the studies of Greenwood [1925], Frost [1933], Berkson and Gage [1952], Fix and Neyman [1951], Boag [1949], Elveback [1958], Armitage [1959], Kaplan and Meier [1958], Dorn [1950], and Littell [1952]. For the material presented in this chapter, reference may be made to Chiang [1961a].

The purpose of this chapter is to adapt the life table methodology and competing risk theory, presented in Appendix II and III, to the special conditions of follow-up studies. Section 2 is concerned with the general type of study which investigates mortality experience without reference to cause of death. The maximum likelihood estimator of the probability of dying is derived, and a method is suggested for computing the observed expectation of life in such studies. Section 3 extends the discussion to follow-up studies with the consideration of competing risks, and presents formulas for the estimators of the net, crude, and partial crude probabilities. The problem of lost cases is treated in Section 4, where a patient's absence is considered as a competing risk. Application of the theoretical matter is illustrated with empirical data of a follow-up study of cervical cancer patients.
2. Estimation of Probability of Survival and Expectation of Life

Consider a follow-up program conducted over a period of $y$ years. A total of $N_0$ patients are admitted to the program at any time during the study period and observed until death or until termination of the study, whichever comes first. The time of admission is taken as the common point of origin for all $N_0$ patients; thus $N_0$ is the number of patients with which the study begins, or the number alive at time zero. The time axis refers to the time of follow-up since admission, and $x$ denotes the exact number of years of follow-up. A constant time interval of one year will be used for simplicity of notation, with the typical interval denoted by $(x, x+1)$, for $x=0,1,\ldots,y-1$. The symbol $p_x$ will be used to denote the probability that a patient alive at time $x$ will survive the interval $(x, x+1)$, and $q_x$ the probability that he will die during the interval, with $p_x+q_x=1$.

2.1. Basic random variables and likelihood functions. For each interval $(x, x+1)$ let $N_x$ be the number of patients alive at the beginning of the interval. Clearly, $N_x$ is also the number of survivors of those who entered the study at least $x$ years before the closing date. The number $N_x$ will decrease as $x$ increases because of deaths and withdrawal of patients due to termination of the study. The decrease in $N_x$ is systematically described below with reference to Table 1.
Table 1

Distribution of $N_x$ patients according to withdrawal status and survival status in the interval $(x, x+1)$

<table>
<thead>
<tr>
<th>Survival status</th>
<th>Withdrawal status in the interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total number of patients</td>
</tr>
<tr>
<td>Total</td>
<td>$N_x$</td>
</tr>
<tr>
<td>Survivors</td>
<td>$s_x + w_x$</td>
</tr>
<tr>
<td>Deaths</td>
<td>$D_x$</td>
</tr>
</tbody>
</table>

* Survivors among those admitted to the study more than $(x+1)$ years before closing date for individual patients.

** Survivors among those admitted to the study less than $(x+1)$ years but more than $x$ years before closing date for individual patients.

The $N_x$ individuals who begin the interval $(x, x+1)$ comprise two mutually exclusive groups differentiated according to their date of entrance into the program. A group of $m_x$ patients who entered the program more than $x+1$ years before the closing date will be observed for the entire interval; a second group of $n_x$ patients who entered the program less than $x+1$ years before its termination is due to withdraw in the interval because the closing date precedes their $(x+1)$th anniversary date. Of the $m_x$ patients $d_x$ will die.
in the interval and \( s_x \) will survive to the end of the interval and become \( N_{x+1} \); of the \( n_x \) patients \( d_x' \) will die before the closing date and \( w_x \) will survive to the closing date of the study. The sum \( d_x + d_x' = D_x \) is the total number of deaths in the interval. Thus \( s_x, d_x, w_x, \) and \( d_x' \) are the basic random variables and will be used to estimate the probability \( p_x \) that a patient alive at \( x \) will survive the interval \( (x, x+1) \), and its complement \( q_x \).

Consider first the group of \( m_x \) individuals each of whom has a constant probability \( p_x \) of surviving and \( q_x = 1 - p_x \) of dying in the interval \( (x, x+1) \). Thus, the random variable \( s_x \) has the binomial distribution:

\[
\begin{align*}
C_1 p_x^{s_x} (1-p_x)^{d_x}
\end{align*}
\]

where \( C_1 \) is the binomial coefficient. The expected number of survivors and the expected number of deaths are given by

\[
\begin{align*}
E(s_x | m_x) &= m_x p_x \\
E(d_x | m_x) &= m_x (1-p_x)
\end{align*}
\]

respectively.

The distribution of the random variables in the group of \( n_x \) patients depends upon the time of withdrawal. A plausible assumption is that the withdrawals take place at random during the
interval \((x, x+1)\). Under this assumption the probability that a patient will survive to the closing date is

\[
-(1-p_x)/\ln p_x ,
\]

(2.3)

which is approximately equal to \(\frac{k_x}{p_x}\), or

\[
-(1-p_x)/\ln p_x = \frac{k_x}{p_x} ,
\]

(2.4)

since the probability \(p_x\) of surviving the interval is almost always large. The quantities on both sides of (2.4) have been computed for selected values of \(p_x\), and the results shown in Table 2 justify the approximation.

<table>
<thead>
<tr>
<th>(p_x)</th>
<th>(\frac{k_x}{p_x})</th>
<th>(-\frac{1-p_x}{\ln p_x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>.70</td>
<td>.837</td>
<td>.841</td>
</tr>
<tr>
<td>.75</td>
<td>.866</td>
<td>.869</td>
</tr>
<tr>
<td>.80</td>
<td>.894</td>
<td>.896</td>
</tr>
<tr>
<td>.85</td>
<td>.922</td>
<td>.923</td>
</tr>
<tr>
<td>.90</td>
<td>.949</td>
<td>.949</td>
</tr>
<tr>
<td>.95</td>
<td>.975</td>
<td>.975</td>
</tr>
</tbody>
</table>
Consequently, $p_x^{b_x}$ is taken as the probability of surviving to the closing date and $(1-p_x^{b_x})$ as the probability of dying before the time of withdrawal. Thus the probability distribution of the random variable $w_x$ in the group of $n_x$ patients due to withdraw is also binomial:

$$C_2 p_x^{b_x} (1-p_x^{b_x})^{d'_x}$$

(2.5)

where $C_2$ is the binomial coefficient. The expected number of survivors and the expected number of deaths are given by

$$E(w_x|n_x) = n_x p_x^{b_x}$$

and

$$E(d_x'|n_x) = n_x (1-p_x^{b_x})$$

(2.6)

respectively.

Since the $N_x$ individuals comprise two independent groups according to their withdrawal status, the likelihood function of all the random variables is the product of the two probability functions (2.1) and (2.5), or

$$L_x = C p_x^{s_x+b_x} x^{d_x} (1-p_x^{b_x})^{x(1-p_x^{b_x})}$$

(2.7)

where $C$ stands for the product of the combinatorial factors in (2.1) and (2.5).
2.2. Maximum likelihood estimators of the probabilities $p_x$ and $q_x$. The maximum likelihood estimators of the probability $p_x$ is a value of $p_x$ at which the function $L_x$ in (2.7) attains a maximum. The estimator is given by

$$
\hat{p}_x = \left[ \frac{-k d_x' + \sqrt{k d_x'^2 + 4(N_x - k n_x)(s_x + k w_x)}}{2(N_x - k n_x)} \right]^2
$$

(2.8)

with the complement

$$
\hat{q}_x = 1 - \hat{p}_x , \quad x = 0, 1, \ldots, y-1.
$$

(2.9)

The maximum likelihood estimator (2.14) is not unbiased, but is consistent in the sense of Fisher. When the random variables $s_x, w_x,$ and $d_x'$ are replaced with their respective expectations as given by (2.5) and (2.9), the resulting expression is identical with the probability $p_x$.

The exact formula for the variance of the estimator $\hat{p}_x$ in (2.8) is unknown, but an approximate formula is stated below for practical applications.

$$
\frac{s^2}{\hat{p}_x \hat{q}_x} = \frac{\hat{p}_x \hat{q}_x}{M_x}
$$

(2.10)

where

$$
M_x = m_x + n_x(1 + \frac{1}{\hat{p}_x})^{-1}
$$

(2.11)
Formula (2.10) is quite similar to the variance of a binomial proportion except that \( M_x \) instead of \( N_x \) is in the denominator. However, \( M_x \) is the more logical choice, since a patient who is to be observed for a fraction of the period \((x, x+1)\) should be weighted less than one who is to be observed for the entire period. According to equation (2.11), the experience of each of the \( m_x \) patients is counted as a whole "trial," whereas the experience of each of the \( n_x \) patients due to withdraw is counted as a fraction \((1+p_{x^*})^{-1}\) of a "trial." The fraction is dependent upon the probability \( p_{x^*} \) of survival. The smaller the probability \( p_{x^*} \), the larger will be the fraction. When \( p_{x^*}=0 \), \( M_x=m_x+n_x \); when \( p_{x^*}=1 \), \( M_x=m_x+n_x \).

2.3. Estimation of survival probability. A life table for follow-up subjects can be readily constructed once \( \hat{p}_x \) and \( \hat{d}_x \) have been determined from (2.8) and (2.9) for each interval of the study. The procedure is the same as for the current life table. Because of their practical importance, we shall consider only the \( x \)-year survival rate and the expectation of life.

The \( x \)-year survival rate is an estimate of the probability that a patient will survive from the time of admission to the \( x \)th anniversary; it is computed from

\[
\hat{p}_{0x} = \hat{p}_0 \hat{p}_1 \ldots \hat{p}_{x-1}, \quad x=1,2,\ldots,y. \quad (2.12)
\]
The sample variance of $\hat{\sigma}^2_{0x}$ has the same form as that given in equation (2.7) of Chapter 3.

$$S^2 = \hat{p}_{0x}^2 \sum_{u=0}^{x-1} \frac{1}{\hat{p}_u} \frac{1}{\hat{p}_{u+1}}.$$ (2.13)

2.4. Estimation of the expectation of life. To avoid confusion in notation, let us denote by $a$ a fixed number and by $\hat{\epsilon}_a$ the observed expectation of life at time $a$ computed from the following formula:

$$\hat{\epsilon}_a = \frac{1}{a} + \hat{p}_a + \hat{p}_a\hat{p}_{a+1} + \cdots + \hat{p}_a\hat{p}_{a+1}\cdots\hat{p}_{a+y-1} + \hat{p}_a\hat{p}_{a+1}\cdots\hat{p}_y + \cdots.$$ (2.14)

In a study covering a period of $y$ years, if no survivors remain from the patients who entered the program in its first year, $\hat{p}_{y-1}$ will be zero, and $\hat{\epsilon}_a$ can be computed from (2.14). However, usually there will be $\hat{w}_{y-1}$ survivors who were admitted in the first year of the program and are still living at the closing date. In such cases (2.8) shows that $\hat{p}_{y-1}$ is greater than zero, and the values of $\hat{p}_y, \hat{p}_{y+1}, \cdots$ are not observed within the time limits of the study. Consequently, $\hat{\epsilon}_a$ cannot be obtained from equation (2.14).

Nevertheless, $\hat{\epsilon}_a$ may be computed with a certain degree of accuracy if $\hat{w}_{y-1}$ is small. Suppose we rewrite equation (2.14) in the form

$$\hat{\epsilon}_a = \frac{1}{a} + \hat{p}_a + \hat{p}_a\hat{p}_{a+1} + \cdots + \hat{p}_a\hat{p}_{a+1}\cdots\hat{p}_{a+y-1} + \hat{p}_a\hat{p}_{a+1}\cdots\hat{p}_y + \cdots,$$ (2.15)

where $\hat{p}_{ay}$ is written for $\hat{p}_a\hat{p}_{a+1}\cdots\hat{p}_{y-1}$. The problem is to determine $\hat{p}_y, \hat{p}_{y+1}, \cdots$ in the last term, since the preceding terms can be computed from the data available.
Consider a typical interval \( (z, z+1) \) beyond time \( y \) with the survival probability of \( p_z \), for \( z = y, y+1, \ldots \). If the force of mortality is constant beyond \( y \), the probability of surviving the interval \( (z, z+1) \) becomes independent of \( z \), or

\[
p_z = p, \quad z = y, y+1, \ldots . \tag{2.16}
\]

Under this assumption, we may replace the last term of (2.15) with \( \hat{p}_{\alpha y} (\hat{p} + \hat{p}^2 + \ldots) \), which converges to \( \hat{p}_{\alpha y} \hat{p} / (1 - \hat{p}) \), or

\[
\hat{p}_{\alpha y} (\hat{p} + \hat{p}^2 + \ldots) = \hat{p}_{\alpha y} \frac{\hat{p}}{1 - \hat{p}} . \tag{2.17}
\]

As a result, we have

\[
\hat{\alpha}_a = \frac{1}{\beta} + \beta \hat{p}_a + \beta \hat{p}_{a+1} + \ldots + \beta \hat{p}_{\alpha a+1} \ldots \hat{p}_{\alpha y-1} + \hat{p}_{\alpha y} \left( \frac{\hat{p}_t}{1 - \hat{p}_t} \right) . \tag{2.18}
\]

Clearly, \( \hat{p} \) may be set equal to \( \hat{p}_{y-1} \) if the force of mortality is assumed to be constant beginning with time \( (y-1) \) instead of time \( y \). In order to have small sample variation, however, the estimate of \( \hat{p} \) should be based on as large a sample as possible. Suppose there exists a time \( t \), for \( t < y \), such that \( \hat{p}_t, \hat{p}_{t+1}, \ldots \) are approximately equal, thus indicating a constant force of mortality after time \( t \). Then, \( \hat{p} \) may be set equal to \( \hat{p}_t \), and we have the formula for the observed expectation of life,

\[
\hat{\alpha}_a = \frac{1}{\beta} + \hat{p}_a + \hat{p}_{\alpha a+1} + \ldots + \hat{p}_{\alpha a+1} \ldots \hat{p}_{\alpha y-1} + \hat{p}_{\alpha y} \left( \frac{\hat{p}_t}{1 - \hat{p}_t} \right) , \tag{2.19}
\]

for \( a = 0, \ldots, y-1 \).
Although formula (2.19) holds for \( a=0, \ldots, y-1 \), it is apparent that the smaller the value of \( a \), the smaller the value of \( \hat{p}_{a y} \).

When \( \hat{p}_{a y} \) is small, the error in assuming a constant force of mortality beyond \( y \) and in the choice of \( \hat{p}_{t} \) will have but little effect on the value of \( \hat{e}_{a} \).

2.5. Sample variance of the observed expectation of life. In Appendix II we prove that the estimated probabilities of surviving any two non-overlapping intervals have a zero covariance; hence, the sample variance of the observed expectation of life may be computed from

\[
\hat{s}_{a}^{2} = \sum_{x>a} \left\{ \frac{3}{\frac{\partial}{\partial p_{x}}} \hat{e}_{a} \frac{2}{\hat{s}_{p_{x}}} \right\}.
\]  

(2.20)

The derivatives, taken at the observed point \( \hat{p}_{x} \), \( x>a \), are given by

\[
\left\{ \frac{3}{\frac{\partial}{\partial p_{x}}} \hat{e}_{a} \right\} = \hat{p}_{a x} \left[ \hat{e}_{x+1} + \frac{1}{2} \right], \quad x \neq t
\]  

(2.21)

where

\[
p_{a x} = p_{a} p_{a+1} \cdots p_{a-1},
\]

and

\[
\left\{ \frac{3}{\frac{\partial}{\partial p_{t}}} \hat{e}_{a} \right\} = \hat{p}_{a t} \left[ \hat{e}_{t+1} + \frac{1}{2} + \frac{\hat{p}_{t y}}{(1-\hat{p}_{t})^{2}} \right], \quad a=t
\]  

(2.22)

For \( t<a \), the factors \( \hat{p}_{a}, \hat{p}_{a+1}, \ldots, \hat{p}_{y-1} \) and \( \hat{p}_{a y} \) in (2.19) do not contain \( \hat{p}_{t} \); hence the derivative

\[
\frac{3}{\frac{\partial}{\partial p_{t}}} \hat{e}_{a} = \frac{3}{\frac{\partial}{\partial p_{t}}} \hat{p}_{a y} \left( \frac{\hat{p}_{t}}{1-\hat{p}_{t}} \right) = \hat{p}_{a y} \frac{1}{(1-\hat{p}_{t})^{2}}
\]  

(2.22a)
Substituting (2.21), (2.22) and (2.22a) in (2.20) gives the sample variance of \( \hat{e}_\alpha \):

\[
S^2_{\hat{e}_\alpha} = \sum_{x=\alpha}^{\gamma-1} \hat{p}_{\alpha x} \left[ \hat{e}_{x+1} + \frac{1}{2} \right]^2 \hat{P}_x^2 + \hat{p}_{\alpha t} \left[ \hat{e}_{t+1} + \frac{1}{2} + \frac{\hat{p}_{\alpha y}}{(1-\hat{p}_t)^2} \right]^2 \hat{p}_t^2, \quad \alpha < t, \tag{2.23}
\]

and

\[
S^2_{\hat{e}_\alpha} = \sum_{x=\alpha}^{\gamma-1} \hat{p}_{\alpha x} \left[ \hat{e}_{x+1} + \frac{1}{2} \right]^2 \hat{P}_x^2 + \frac{\hat{p}_{\alpha y}}{(1-\hat{p}_t)^4} \hat{p}_t^2, \quad \alpha > t. \tag{2.24}
\]

The value of \( \hat{p}_x \) and the sample variance of \( \hat{p}_x \) are obtained from formulas (2.8) and (2.10), respectively.

When the first term in formula (2.23) or (2.24) is taken out of the summation sign, we have a recursive equation

\[
S^2_{\hat{e}_\alpha} = \left[ \hat{e}_{\alpha+1} + \frac{1}{2} \right]^2 \hat{P}_\alpha^2 + \frac{\hat{p}_{\alpha y}}{\hat{e}_{\alpha+1}} \hat{p}_\alpha^2, \quad \text{for } \alpha \neq t. \tag{2.25}
\]

Therefore, the variance of \( \hat{e}_\alpha \) may be computed successively beginning with the largest value of \( \alpha \).

### 2.6 An example of life table construction for a follow-up population.

Application of the methods developed in this section is illustrated with data collected by the Tumor Registry of the California State Department of Public Health. The material selected consists of 5,982 white female patients admitted to certain California hospitals and clinics between January 1, 1942, and December 31, 1954, with a diagnosis of cervical cancer.

For the purpose of this illustration, the closing date is December 31, 1954,
and the date of entrance to follow-up for each patient is the date of hospital admission. Each patient was observed until death or until the closing date, whichever came first.

The first step is to construct a table similar to Table 3, showing the survival experience of the patients grouped according to their withdrawal status for each time period of follow-up. The interval length selected (column 1) will depend upon the nature of the investigation; generally a fixed length of one year is used. The total number of patients admitted to the study is entered as $N_0$ in the first line of column 2, which is 5,982. Among them there were $m_0 = 5,317$ patients (column 3), observed for the entire interval (0,1). Of the $m_0$ patients, $s_0 = 4,030$ (column 4) survived to their first anniversary and $d_0 = 1,287$ (column 5) died during the first year of follow-up. In addition, there were $n_0 = 565$ (column 6) patients due to withdraw in the interval (0,1), of which $w_0 = 576$ (column 7) survived to the closing date and $d_0 = 89$ (column 8) died before the closing date. The second interval began with the $s_0 = 4,030$ survivors from the first interval, which is entered as $N_1$ in column 2 of line 2. The $N_1$ patients were again divided successively by withdrawal and survival status. Of the $N_1$ patients, $m_1 = 3,489$ (column 3) were the survivors of those admitted prior to January 1, 1953, and hence were observed for the entire interval (1, 2); $n_1 = 541$ (column 8) were the survivors of those admitted during the year 1953 and hence were due to withdraw during the interval. At the beginning of the final interval (12, 13) there were $N_{12} = 72$ survivors of the patients admitted in 1942; all were due to withdraw during the last interval, or $n_{12} = 72$ (last line, column 8). Of the 72 patients, $w_{12} = 72$ (column 7) were alive at the closing date. This means that $\hat{p}_{12}$ is greater than zero, and $\hat{p}_z$ for $z > 12$ cannot be observed.
Table 3
SURVIVAL EXPERIENCE FOLLOWING DIAGNOSIS OF CANCER OF THE CERVIX UTERI
CASES INITIALLY DIAGNOSED 1942-1954
CALIFORNIA, U.S.A.

<table>
<thead>
<tr>
<th>Interval since diagnosis (in years)</th>
<th>Number living at beginning of interval (x, x+1)</th>
<th>Number to be observed for entire interval (x, x+1)*</th>
<th>Number due for withdrawal in interval (x, x+1)**</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N_x</td>
<td>m_x</td>
<td>s_x</td>
</tr>
<tr>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>5982</td>
<td>5317</td>
<td>4030</td>
</tr>
<tr>
<td>1-2</td>
<td>4030</td>
<td>3489</td>
<td>2845</td>
</tr>
<tr>
<td>2-3</td>
<td>2845</td>
<td>2367</td>
<td>2117</td>
</tr>
<tr>
<td>3-4</td>
<td>2117</td>
<td>1724</td>
<td>1573</td>
</tr>
<tr>
<td>4-5</td>
<td>1573</td>
<td>1263</td>
<td>1176</td>
</tr>
<tr>
<td>5-6</td>
<td>1176</td>
<td>918</td>
<td>861</td>
</tr>
<tr>
<td>6-7</td>
<td>861</td>
<td>692</td>
<td>660</td>
</tr>
<tr>
<td>7-8</td>
<td>660</td>
<td>496</td>
<td>474</td>
</tr>
<tr>
<td>8-9</td>
<td>474</td>
<td>356</td>
<td>344</td>
</tr>
<tr>
<td>9-10</td>
<td>344</td>
<td>256</td>
<td>245</td>
</tr>
<tr>
<td>10-11</td>
<td>245</td>
<td>164</td>
<td>158</td>
</tr>
<tr>
<td>11-12</td>
<td>158</td>
<td>76</td>
<td>72</td>
</tr>
<tr>
<td>12-13</td>
<td>72</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Survivors of those admitted more than x+1 years prior to closing date.
**Survivors of those admitted between x and x+1 years prior to closing date.

Source: California Tumor Registry, Department of Public Health, State of California
This material has been used to construct a life table for the cervical cancer patients. The steps involved are similar to those described in the construction of current life tables in Chapter 3. For easy reference, but at the expense of repetition, they are stated below:

1. $p_x$ and $q_x$. For each interval $(x, x+1)$, use formulas (2.8) and (2.9) of this chapter to compute $p_x$ and $q_x$.

2. $d_x$ and $l_x$. Assume $e_0 = 100,000$, use $q_0, q_1, \ldots$ to obtain $d_x$ and $l_x$ from

$$d_x = l_x q_x \quad \text{and} \quad l_{x+1} = l_x - d_x$$

for $x = 0, 1, \ldots, 12$.

3. $a_x$ and $L_x$. The fraction of last year of life is assumed to be $a_x = .5$, which is quite appropriate for such studies.

The quantity $L_x$ is computed from

$$L_x = l_{x+1} + a_x d_x$$

or since $a_x = .5$ and $d_x = l_x - l_{x+1}$,

$$L_x = \frac{1}{2}(l_x + l_{x+1}), \quad \text{for } x = 0, 1, \ldots, 12.$$

4. $T_x$ and $e_x$ beyond the observation period. Information derived from a follow-up study is incomplete for the construction of a life table inasmuch as it is limited to the study period (13 years in this example). Therefore, some device needs to be developed for the computation of $e_x$ beyond the last year of study, that is $e_{13}$ in the present case. Here we make use of equation (2.19) of this chapter and
write
\[ \hat{e}_{13} = \frac{1}{\hat{p}_t} + \frac{\hat{p}_t}{1 - \hat{p}_t} \]  
(2.19 a)

Estimating \( p_t \) with \( \hat{p}_{11} \),
\[ \hat{p}_t = \hat{p}_{11} = 1 - \hat{q}_{11} = 1 - .05106 = .94894 \]
gives required values
\[ \hat{e}_{13} = \frac{1}{\hat{p}_t} + \frac{.94894}{1 - .94894} = 19.0848 \]

Using this figure we compute
\[ T_{13} = \ell_{13} \hat{e}_{13} = 34,277 \times 19.0848 = 654,170 \]  

(5) \( T_x \) and \( \hat{e}_x \). The quantities \( \ell_x \) and \( \hat{e}_x \) for other intervals can be obtained by simple computations. For example,

\[ T_{12} = \ell_{12} + T_{13} \]

In general
\[ T_x = \ell_x + T_{x+1} \quad \text{for } x = 0, \ldots, 12, \]

and \( \hat{e}_x \) (except for \( \hat{e}_{13} \)) is computed from 3/

\[ \hat{e}_x = \frac{T_x}{\ell_x} \quad \text{for } x = 0, 1, \ldots, 12. \]

The results of the computations are given in Table 4.

For comparison between survival experience of different study groups or for making other statistical inferences, we computed the standard deviations of the survival rate [eq. (2.13)], of probability of death [eq. (2.10)], and of the expectation of life [eqs. (2.23), (2.24) and (2.25)] for each \( x \).

Numerical values of the standard errors and the main life table functions are shown in Table 5.
For example, at \( x=2 \), the calculations for \( S_{\alpha} \) were as follows:

\[
S_{\alpha}^2 = \frac{q_x^2 (1-q_x^2)}{M_x} \quad \text{where} \quad M_x = m_x + n \left(1 + p_x^2\right)^{-1}
\]

\[
M_2 = m_2 + n_2 + (1 + p_2^2)^{-1} = 2367 + 478(1 + .8967^2)^{-1} = 2,612.495
\]

\[
S_{\alpha}^2 = \frac{q_2^2 (1-q_2^2)}{M_2} = \frac{.10303(.89697)}{2612.495} = .00003537
\]

\[
S_{\alpha} = .00003537 = .00595 = S_{\alpha_2}
\]

To calculate \( S_{\alpha} \) from \( S_{\alpha}^2 \):

\[
S_{\alpha}^2 = S_{\alpha_2}^2 \left(1 - \frac{e}{p_0^2}\right) \quad \text{where} \quad p_0 = p_0^2 + e
\]

\[
S_{\alpha} = \sqrt{.00003537} = .00595 = S_{\alpha_2}
\]

To calculate \( S_{e_3} \) from \( S_{e_3}^2 \):

\[
S_{e_3}^2 = p_3^2 S_{e_3}^2 + \left[e_4 + .5\right]^2 S_{e_4}^2
\]

\[
S_{e_3} = \sqrt{20.8584} = 4.567
\]
Table 4
LIFE TABLE OF PATIENTS DIAGNOSED AS HAVING CANCER OF THE CERVIX UTERI
CASES INITIALLY DIAGNOSED 1942 - 1954
CALIFORNIA, U.S.A.

<table>
<thead>
<tr>
<th>Interval since diagnosis (years)</th>
<th>Number living at time</th>
<th>Probability of dying in interval $l_x$</th>
<th>Number dying in interval $d_x$</th>
<th>Fraction of last year of life $a_x$</th>
<th>Number of years lived in interval $L_x$</th>
<th>Number of years lived beyond interval $T_x$</th>
<th>Observed Expectation of life at $x$ $\tilde{e}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>100,000</td>
<td>.24254</td>
<td>24,254</td>
<td>.5</td>
<td>87,783</td>
<td>1,289,575</td>
<td>12.90</td>
</tr>
<tr>
<td>1-2</td>
<td>75,746</td>
<td>.18143</td>
<td>13,743</td>
<td>.5</td>
<td>68,875</td>
<td>1,201,702</td>
<td>15.86</td>
</tr>
<tr>
<td>2-3</td>
<td>62,003</td>
<td>.10303</td>
<td>6,388</td>
<td>.5</td>
<td>58,809</td>
<td>1,132,827</td>
<td>18.27</td>
</tr>
<tr>
<td>3-4</td>
<td>55,615</td>
<td>.08376</td>
<td>4,770</td>
<td>.5</td>
<td>53,230</td>
<td>1,074,018</td>
<td>19.31</td>
</tr>
<tr>
<td>4-5</td>
<td>50,845</td>
<td>.06413</td>
<td>3,261</td>
<td>.5</td>
<td>49,215</td>
<td>1,020,788</td>
<td>20.08</td>
</tr>
<tr>
<td>5-6</td>
<td>47,584</td>
<td>.05820</td>
<td>2,769</td>
<td>.5</td>
<td>46,200</td>
<td>971,573</td>
<td>20.42</td>
</tr>
<tr>
<td>6-7</td>
<td>44,815</td>
<td>.04376</td>
<td>1,961</td>
<td>.5</td>
<td>43,835</td>
<td>925,373</td>
<td>20.65</td>
</tr>
<tr>
<td>7-8</td>
<td>42,854</td>
<td>.04320</td>
<td>1,851</td>
<td>.5</td>
<td>41,929</td>
<td>881,538</td>
<td>20.57</td>
</tr>
<tr>
<td>8-9</td>
<td>41,003</td>
<td>.03369</td>
<td>1,381</td>
<td>.5</td>
<td>40,313</td>
<td>799,296</td>
<td>20.48</td>
</tr>
<tr>
<td>9-10</td>
<td>39,622</td>
<td>.04365</td>
<td>1,844</td>
<td>.5</td>
<td>38,700</td>
<td>760,396</td>
<td>20.17</td>
</tr>
<tr>
<td>10-11</td>
<td>37,778</td>
<td>.04385</td>
<td>1,657</td>
<td>.5</td>
<td>36,950</td>
<td>723,646</td>
<td>20.03</td>
</tr>
<tr>
<td>11-12</td>
<td>36,121</td>
<td>.05106</td>
<td>1,844</td>
<td>.5</td>
<td>35,199</td>
<td>688,447</td>
<td>20.08</td>
</tr>
<tr>
<td>12-13</td>
<td>34,277</td>
<td>.00000</td>
<td>0</td>
<td>.5</td>
<td>34,277</td>
<td>654,170</td>
<td>19.08*</td>
</tr>
</tbody>
</table>

* For computation of $e_{13}$ and $T_{13}$ see text (2.19a)
Table 5
SURVIVAL EXPERIENCE AFTER DIAGNOSIS OF CANCER OF THE CERVIX UTERI
CASES INITIALLY DIAGNOSED 1942-1954
CALIFORNIA, U.S.A.
THE MAIN LIFE TABLE FUNCTIONS AND THEIR STANDARD ERRORS

<table>
<thead>
<tr>
<th>Interval since diagnoses (years)</th>
<th>x-year survival rate $\hat{P}_{0x}$</th>
<th>Estimated probability of death in interval $(x, x+1)$</th>
<th>Observed Expectation of life at $x$ $^{a/}$</th>
<th>$S_{e_x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, x+1)$</td>
<td>1000 $\hat{P}_{0x}$</td>
<td>1000 $S_{\hat{P}_{0x}}$</td>
<td>1000 $\hat{q}_x$</td>
<td>1000 $S_{\hat{q}_x}$</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>0-1</td>
<td>1000.00</td>
<td>0.00</td>
<td>242.54</td>
<td>5.69</td>
</tr>
<tr>
<td>1-2</td>
<td>757.46</td>
<td>5.80</td>
<td>181.43</td>
<td>6.26</td>
</tr>
<tr>
<td>2-3</td>
<td>620.03</td>
<td>6.65</td>
<td>103.03</td>
<td>5.95</td>
</tr>
<tr>
<td>3-4</td>
<td>556.15</td>
<td>7.01</td>
<td>85.76</td>
<td>6.38</td>
</tr>
<tr>
<td>4-5</td>
<td>508.45</td>
<td>7.33</td>
<td>64.13</td>
<td>6.50</td>
</tr>
<tr>
<td>5-6</td>
<td>475.84</td>
<td>7.61</td>
<td>58.20</td>
<td>7.23</td>
</tr>
<tr>
<td>6-7</td>
<td>448.15</td>
<td>7.95</td>
<td>43.76</td>
<td>7.34</td>
</tr>
<tr>
<td>7-8</td>
<td>428.54</td>
<td>8.29</td>
<td>43.20</td>
<td>8.45</td>
</tr>
<tr>
<td>8-9</td>
<td>410.03</td>
<td>8.71</td>
<td>33.69</td>
<td>8.85</td>
</tr>
<tr>
<td>9-10</td>
<td>396.22</td>
<td>9.17</td>
<td>46.55</td>
<td>12.15</td>
</tr>
<tr>
<td>10-11</td>
<td>377.78</td>
<td>9.98</td>
<td>43.85</td>
<td>14.30</td>
</tr>
<tr>
<td>11-12</td>
<td>361.21</td>
<td>10.97</td>
<td>51.06</td>
<td>20.30</td>
</tr>
<tr>
<td>12-13</td>
<td>342.77</td>
<td>12.73</td>
<td>00.00</td>
<td>20.08</td>
</tr>
<tr>
<td>13</td>
<td>342.77</td>
<td>12.73</td>
<td>-----</td>
<td>-----</td>
</tr>
</tbody>
</table>

Source: California Tumor Registry, Department of Public Health, State of California, U.S.A.

$a/_{t = 11}$
3. Consideration of Competing Risks

Most follow-up studies are conducted to determine the survival rates of patients affected with a specific disease. These patients are also exposed to other risks of death from which some of them may eventually die. In a study determining the effectiveness of radiation as a treatment for cancer, for example, some patients may die from heart disease. In such cases, the theory of competing risks is indispensible, and the crude, net, and partial crude probabilities all play important roles.

Let us assume, as in Appendix III, that \( r \) risks, denoted by \( R_1, \ldots, R_r \), are acting simultaneously on each patient in the study. For risk \( R_\delta \) there is a corresponding force of mortality \( \mu(t;\delta) \), \( \delta=1,\ldots,r \), and the sum

\[
\mu(t;1) + \cdots + \mu(t;r) = \mu(t)
\]  

(3.1)

is the total force of mortality. Within the time interval \((x, x+1)\) we assume a constant force of mortality for each risk, \( \mu(t;\delta) = \mu(x;\delta) \), which depends only on the interval \((x, x+1)\) and the risk \( R_\delta \); for all risks, \( \mu(t) = \mu(x) \) for \( x < t \leq x+1 \).

Consider a subinterval \((x, x+t)\), and let \( Q_{x\delta}(t) \) be the crude probability that an individual alive at time \( x \) will die prior to \( x+t \), \( 0 < t \leq 1 \), from \( R_\delta \) in the presence of all other risks in the population. It follows directly from equation (2.8) in Mathematical Appendix III that

\[
Q_{x\delta}(t) = \frac{\mu(x;\delta)}{\mu(x)} \left[ 1 - p_x(t) \right], \quad 0 < t \leq 1; \quad \delta=1,\ldots,r.
\]  

(3.2)
From (3.1) we see that the sum of the crude probabilities in (3.2) is equal to the complement of $p_x(t)$, or

$$Q_{x1}(t) + \cdots + Q_{xr}(t) + p_x(t) = 1, \quad 0 < t \leq 1. \tag{3.3}$$

For $t=1$, we abbreviate $Q_{x\delta}(1)$ to $Q_{x\delta}'$, etc. When $t = \frac{1}{2}$, we have the subinterval $(x, x+\frac{1}{2})$ and the corresponding crude probabilities

$$Q_{x\delta}(\frac{1}{2}) = \frac{\nu(x; \delta)}{\nu(x)} \left[ 1 - p_x^{\frac{1}{2}} \right] = Q_x^{0\delta} \left[ 1 + p_x^{\frac{1}{2}} \right]^{-1}, \delta = 1, \cdots, r. \tag{3.4}$$

Equation (3.3) implies that

$$Q_{x1} \left[ 1 + p_x^{\frac{1}{2}} \right]^{-1} + \cdots + Q_{xr} \left[ 1 + p_x^{\frac{1}{2}} \right]^{-1} + p_x^{\frac{1}{2}} = 1, \quad x = 0, 1, \cdots, y-1. \tag{3.5}$$

The net and partial crude probabilities may be computed from the following approximate relations. The corresponding exact formulas are given in Section 2, Appendix III. The net probability of death in interval $(x, x+1)$ when $\gamma$ is the only risk operating in a population is given by

$$q_{x\delta} = q_x^{0\delta} \left[ 1 + p_x^{\frac{1}{2}} \right] + \frac{1}{\delta} \left( q_x^{0\delta} - q_x^{1\delta} \right) (2q_x^{0\delta} - q_x^{1\delta}) \tag{3.6}$$

the net probability of death if $\gamma$ is eliminated as a risk of death is given by
and the partial crude probability by

\[ \eta_{x,t} = (\eta_{x+\xi},1) + \frac{1}{\gamma_{x+\xi}} + \frac{1}{\delta} \gamma_{x+\xi} (\eta_{x+\xi},1) \] \quad \delta = 1, \ldots, r, \quad (3.7)

for \( r = 2, \ldots, R; x = 0, 1, \ldots, n-1 \).

Our immediate problem is to estimate \( Q_{x,t} \), \( p_x \), and \( q_x \).

3.1. Basic random variables and likelihood functions. Identification of the random variables in the present case follows directly from the discussion in Section 2.1, except that deaths are further divided by cause, as shown in Table 6.
Table 6
Distribution of $N_x$ patients according to withdrawal status, survival status, and cause of death in the interval ($x$, $x+1$)

<table>
<thead>
<tr>
<th>Withdrawal status in the interval</th>
<th>Total number of patients</th>
<th>Number to be observed for the entire interval*</th>
<th>Number due to withdraw during the interval**</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>$N_x$</td>
<td>$m_x$</td>
<td>$n_x$</td>
</tr>
<tr>
<td>Survivors</td>
<td>$s_x + w_x$</td>
<td>$s_x$</td>
<td>$v_x$</td>
</tr>
<tr>
<td>Deaths, all causes</td>
<td>$D_x$</td>
<td>$d_x$</td>
<td>$d'_x$</td>
</tr>
</tbody>
</table>

Deaths due to cause

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$D_{x1}$</th>
<th>$d_{x1}$</th>
<th>$d'_{x1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>$R_r$</td>
<td>$D_{xr}$</td>
<td>$d_{xr}$</td>
<td>$d'_{xr}$</td>
</tr>
</tbody>
</table>

* Survivors among those admitted to the study more than ($x+1$) years before closing date.

** Survivors among those admitted to the study less than ($x+1$) years but more than $x$ years before closing date.

The $m_x$ patients to be observed for the entire interval ($x$, $x+1$) will be divided into $r+1$ mutually exclusive groups, with $s_x$ surviving the interval and $d_{x\delta}$ dying from cause $R_{\delta}$ in the interval, $\delta=1,\ldots,r$. Since the sum of the corresponding probabilities is equal to unity.
(eq. (3.3) the random variables $s_x, d_{x1}, \ldots, d_{xr}$ have a multinomial distribution:

$$C_1 p_x q_{x1} \ldots q_{xr}, \quad (3.10)$$

where $s_x + d_{x1} + \cdots + d_{xr} = m_x$, and $C_1$ is the combinatorial factor. The expected numbers are given by

$$E(s_x | m_x) = m_x p_x \quad \text{and} \quad E(d_{x0} | m_x) = m_x q_{x0}, \quad (3.11)$$

respectively.

In the group of $n_x$ patients due to withdraw in interval $(x, x+1)$, $w_x$ will be alive at the closing date of the study and $d_{x0}$ will die from $R_0$ before the closing date. Each of the $n_x$ individuals has the survival probability $p_x^+ [\text{Cf. eq. (2.4)}]$ and the probability of dying from risk $R_0$ before the closing date

$$Q_{x0}^+ (\delta) = Q_{x0} (1 + p_x^+)^{-1}, \quad \delta = 1, \ldots, r. \quad (3.12)$$

Since $p_x^+$ and the probabilities in (3.12) add to unity, as shown in (3.5), the random variables $w_x, d_{x1}, \ldots, d_{xr}$ also have a multinomial distribution:

$$C_2 p_x^+ \sum_{\delta=1}^r \left[ Q_{x0} (1 + p_x^+)^{-1} \right] d_{x0}^\delta, \quad (3.13)$$

where $w_x + d_{x1}^\delta + \cdots + d_{xr}^\delta = n_x$, and $C_2$ is a combinatorial factor.
The expected numbers are

\[ E(w_x^+|n_x) = n_x p_x^{1/2} \quad \text{and} \quad E(d_x^+|n_x) = n_x Q_x (1+p_x^{1/2})^{-1} \] (3.14)

respectively. Because of the independence of the two groups, the likelihood function of all the random variables in Table 6 is the product of (3.10) and (3.13):

\[ L_x = c p_x^{s+1/2} \prod_{\delta=1}^r Q_x \sum_{x} d_x^0 \left[ Q_x (1+p_x^{1/2})^{-1} \right]^{d_x^0} \] (3.15)

where \( c \) stands for the product of the combinatorial factors in (3.10) and (3.13). Equation (3.15) may be simplified to give the final form of the likelihood function

\[ L_x = c p_x^{s+1/2} \prod_{\delta=1}^r Q_x \sum_{x} d_x^0 \left[ Q_x (1+p_x^{1/2})^{-1} \right]^{d_x^0} \] (3.16)

3.2 Estimation of crude, net, and partial crude probabilities. We again use the maximum likelihood principle to obtain the estimators of the probabilities \( p_x, Q_{xl}, \ldots, Q_{xr} \). The estimator of \( p_x \) is the same as that obtained in Section 2; namely

\[ \hat{p}_x = \left[ \frac{-d_x^0 + \sqrt{d_x^0^2 + 4(N_x - \sum_{x} d_x^0) (s_h + w)} - 2(N_x - \sum_{x} d_x^0)}{2(N_x - \sum_{x} d_x^0)} \right] , \quad x=0,1,\ldots,y-1. \] (3.17)

Therefore \( \hat{q}_x = (1-\hat{p}_x) \) also will have the same values as that in Section 2. The estimators of the crude probabilities are given by

\[ \hat{Q}_{x0}^\delta = \frac{D_{x0}^\delta}{D_x} \hat{q}_x \quad , \quad \delta=1,2,\ldots,r, \quad x=0,1,\ldots,y-1. \] (3.18)
We now use (3.17) and (3.19) in formulas (3.6) to (3.9) to obtain the following estimators of the net and partial crude probabilities:

\[
\hat{\theta}_{w} = \hat{\theta}_{w} \left[ 1 + \frac{1}{n} (\hat{\theta}_{w} - \hat{\theta}_{w}) + \frac{1}{n} (\hat{\theta}_{w} - \hat{\theta}_{w}) (2 \hat{\theta}_{w} - \hat{\theta}_{w}) \right]
\]

(3.19)

\[
\hat{\theta}_{w,\delta} = (\hat{\theta}_{w} - \hat{\theta}_{w}) \left[ 1 + \frac{1}{n} \hat{\theta}_{w} + \frac{1}{n} \hat{\theta}_{w} (\hat{\theta}_{w} + \hat{\theta}_{w}) \right] \quad , \quad \delta = 1, \ldots, r,
\]

(3.20)

and

\[
\hat{\theta}_{x,\delta} = \hat{\theta}_{x} \left[ 1 + \frac{1}{n} \hat{\theta}_{x} + \frac{1}{n} \hat{\theta}_{x} (\hat{\theta}_{x} + \hat{\theta}_{x}) \right] \quad , \quad \delta = 1, \ldots, r
\]

(3.21)

x = 1, \ldots, v-1.

These are also maximum likelihood estimators and consistent in Fisher's sense. Consider for example the estimator \( \hat{\theta}_{x,1} \) in formula (3.21) of the partial crude probability. We have seen in Section 2 that \( \hat{\theta}_{x} \) is consistent in Fisher's sense. Then the other random variables are replaced with the corresponding expectations, the right side of (3.21) may be simplified to \( \hat{\theta}_{x,1} \) given in (3.8), proving the consistency.
3.3. An Example. The survival experience of cervical cancer patients presented in Table 3 in Section 2.6 is used once again to illustrate the application of the theory in this section. For easy reference, the cervical cancer patients data is reproduced in Table 7, except that the number of deaths, \( d_x \) and \( d'_x \) are further divided according to cause. In this example only two causes are considered, cancer of the cervix and other causes. Therefore, we have for each interval \((x, x+1)\),

\[
d_x = d_{x1} + d_{x2} \quad \text{and} \quad d'_x = d'_{x1} + d'_{x2}
\]

During the first year of follow-up, for example, there were \( d_0 = 1,287 \) deaths occurring among those to be observed for the entire interval, and \( d'_0 = 89 \) deaths among those due for withdrawal in the interval. These numbers are divided by causes:

\[
1,287 = 1,105 + 182 \quad \text{and} \quad 89 = 70 + 19
\]

With the numerical values of the probability of survival \( \hat{p}_x \) and the probability of dying \( \hat{q}_x \) obtained in Section 2, simple application of formulas (3.18) and (3.19) yield the crude probability \( \hat{Q}_{x0} \) and the net probability \( \hat{q}_{x0} \). Since only two causes of death are studied, the probability \( \hat{q}_{x2} \) is equal to \( \hat{q}_{x1} \); the probability of dying when cancer of the cervix uteri is eliminated is the same as the probability of dying from other causes when the other causes are the only causes acting. Table 8 shows the estimated probability of surviving each interval, and the crude and net probabilities of death from cancer of the cervix uteri \( (R_1) \) and all other causes of death \( (R_2) \).
### Table 7

**Survival Experience Following Diagnosis of Cancer of the Cervix Uteri**

**Cases Initially Diagnosed 1942-1954**

**California, U.S.A.**

<table>
<thead>
<tr>
<th>Interval since diagnosis (in years)</th>
<th>Number living at beginning of interval ((x, x+1))</th>
<th>Number to be observed during the entire interval ((x, x+1))*</th>
<th>Number due for withdrawal in interval ((x, x+1))</th>
<th>Number dying before withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x, x+1))</td>
<td>(N_x)</td>
<td>(m_x)</td>
<td>(s_x)</td>
<td>(d_x)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>0-1</td>
<td>5982</td>
<td>5317</td>
<td>4030</td>
<td>1287</td>
</tr>
<tr>
<td>1-2</td>
<td>4030</td>
<td>3489</td>
<td>2845</td>
<td>1287</td>
</tr>
<tr>
<td>2-3</td>
<td>2845</td>
<td>2367</td>
<td>2117</td>
<td>1287</td>
</tr>
<tr>
<td>3-4</td>
<td>1724</td>
<td>1861</td>
<td>1573</td>
<td>1176</td>
</tr>
<tr>
<td>4-5</td>
<td>1176</td>
<td>818</td>
<td>861</td>
<td>57</td>
</tr>
<tr>
<td>5-6</td>
<td>861</td>
<td>692</td>
<td>660</td>
<td>32</td>
</tr>
<tr>
<td>6-7</td>
<td>660</td>
<td>496</td>
<td>474</td>
<td>22</td>
</tr>
<tr>
<td>7-8</td>
<td>474</td>
<td>356</td>
<td>344</td>
<td>12</td>
</tr>
<tr>
<td>8-9</td>
<td>344</td>
<td>256</td>
<td>245</td>
<td>11</td>
</tr>
<tr>
<td>9-10</td>
<td>245</td>
<td>164</td>
<td>158</td>
<td>6</td>
</tr>
<tr>
<td>10-11</td>
<td>158</td>
<td>76</td>
<td>72</td>
<td>4</td>
</tr>
<tr>
<td>11-12</td>
<td>72</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

---

* Survivors of those admitted more than \(x+1\) years prior to closing date.

** Survivors of those admitted between \(x\) and \(x+1\) years prior to closing date.

Source: California Tumor Registry, Department of Public Health, State of California, U.S.A.
### Table 8

**SURVIVAL EXPERIENCE AFTER DIAGNOSIS OF CANCER OF THE CERVIX UTERI**

**CASES INITIALLY DIAGNOSED 1942-1954**

**CALIFORNIA, U. S. A.**

**ESTIMATED CRUDE AND NET PROBABILITIES OF DEATH FROM CANCER OF THE CERVIX UTERI AND FROM OTHER CAUSES**

<table>
<thead>
<tr>
<th>Interval since diagnosis (years)</th>
<th>Probability of surviving interval (x, x+1)</th>
<th>Crude probabilities of death in interval (x, x+1) from Cervix cancer</th>
<th>Other causes</th>
<th>Cervix Cancer Acting Alone</th>
<th>Cervix Cancer Eliminated</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, x+1)</td>
<td>1000 $\hat{p}_x$</td>
<td>1000 $\hat{Q}_{x1}$</td>
<td>1000 $\hat{Q}_{x2}$</td>
<td>1000 $\hat{q}_{x1}$</td>
<td>1000 $\hat{q}_{x2}$</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
</tr>
<tr>
<td>0-1</td>
<td>757.46</td>
<td>207.11</td>
<td>35.43</td>
<td>211.17</td>
<td>39.77</td>
</tr>
<tr>
<td>1-2</td>
<td>818.57</td>
<td>155.97</td>
<td>25.46</td>
<td>158.11</td>
<td>27.71</td>
</tr>
<tr>
<td>2-3</td>
<td>896.97</td>
<td>84.65</td>
<td>18.38</td>
<td>85.46</td>
<td>19.22</td>
</tr>
<tr>
<td>3-4</td>
<td>914.24</td>
<td>62.89</td>
<td>22.87</td>
<td>63.63</td>
<td>23.63</td>
</tr>
<tr>
<td>4-5</td>
<td>935.87</td>
<td>44.40</td>
<td>19.73</td>
<td>44.85</td>
<td>20.19</td>
</tr>
<tr>
<td>5-6</td>
<td>941.80</td>
<td>25.76</td>
<td>32.44</td>
<td>26.19</td>
<td>32.87</td>
</tr>
<tr>
<td>6-7</td>
<td>956.24</td>
<td>23.17</td>
<td>20.59</td>
<td>23.41</td>
<td>20.84</td>
</tr>
<tr>
<td>7-8</td>
<td>956.80</td>
<td>22.47</td>
<td>20.73</td>
<td>22.70</td>
<td>20.97</td>
</tr>
<tr>
<td>8-9</td>
<td>966.31</td>
<td>14.44</td>
<td>19.25</td>
<td>14.58</td>
<td>19.39</td>
</tr>
<tr>
<td>9-10</td>
<td>953.45</td>
<td>29.93</td>
<td>16.62</td>
<td>30.18</td>
<td>16.88</td>
</tr>
<tr>
<td>10-11</td>
<td>956.15</td>
<td>24.36</td>
<td>19.49</td>
<td>24.60</td>
<td>19.73</td>
</tr>
<tr>
<td>11-12</td>
<td>948.94</td>
<td>17.02</td>
<td>34.04</td>
<td>17.32</td>
<td>34.34</td>
</tr>
<tr>
<td>12-13</td>
<td>1000.00</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>

**Source:** California Tumor Registry, Department of Public Health, State of California, U.S.A.
Every patient in a medical follow-up is exposed not only to the risk of dying, but also to the risk of being lost to the study because of follow-up failure. Untraceable patients have caused difficulties in determining survival rates, as have patients withdrawing due to the termination of a study. However, lost cases and withdrawals belong to entirely different categories. In a group of $N_x$ patients beginning the interval $(x, x+1)$, for example, everyone is exposed to the risk of being lost, but only $n_x$ patients are subject to withdrawal in the interval. Therefore, it is incorrect to treat lost cases and withdrawals equally in estimating probabilities of survival or death. For the purpose of determining the probability of dying from a specific cause, patients lost due to follow-up failure are not different from those dying of causes unrelated to the study. Being lost, therefore, should be considered as a competing risk, and the survival experience of lost cases should be evaluated by using the methods discussed in the preceding section. In this approach to the problem all formulas in Section 3 will remain intact, the solution requiring only a different interpretation of the symbols.

Suppose we let $R_\tau$ denote the risk of being lost; for the time element $(\tau, \tau+\Delta)$ in the interval $(x, x+1)$ let

$$\mu(x;\tau)\Delta + o(\Delta) = \Pr\{\text{a patient will be lost to the study in}$$

$$\text{(}\tau, \tau+\Delta) \text{ due to follow-up failure}\}, \ x<\tau<x+1. \ (4.1)$$
The following are a few examples of the new interpretation:

\[ P_x = \Pr\{a \text{ patient alive at time } x \text{ will remain alive and under observation at time } x+1\} \]  \hspace{1cm} (4.2)

\[ q_x = 1 - p_x \]
\[ = \Pr\{a \text{ patient alive at time } x \text{ will either die or be lost to the study due to follow-up failure in interval (}x, x+1)\} \]  \hspace{1cm} (4.3)

\[ Q_{xr} = \Pr\{a \text{ patient alive at time } x \text{ will be lost to the study in } (x, x+1)\} \]  \hspace{1cm} (4.4)

\[ q_{x.r} = \Pr\{a \text{ patient alive at time } x \text{ will die in interval } (x, x+1) \text{ if the risk } R_r \text{ of being lost is eliminated}\} \]  \hspace{1cm} (4.5)

\[ 1 - q_{x.r} = \Pr\{a \text{ patient alive at } x \text{ will survive to time } x+1 \text{ if the risk } R_r \text{ of being lost is eliminated}\} \]  \hspace{1cm} (4.6)

\[ Q_{x.r} = \Pr\{a \text{ patient alive at } x \text{ will die in } (x, x+1) \text{ from risk } R_\delta \text{ if the risk } R_r \text{ of being lost is eliminated}\} \]  \hspace{1cm} (4.7)
The probabilities in (4.5), (4.6), and (4.7) are equivalent to $q_x$, $p_x$, and $Q_x$, respectively, if there is no risk of being lost.

The symbol $d_{xr}$ in Table 6 now stands for the number of lost cases among the $m_x$ patients and $d'_{xr}$ for the number of lost cases among the $n_x$ patients; the sum $D_{xr} = d_{xr} + d'_{xr}$ is the total number of cases lost in the interval. The probabilities in (4.2) through (4.7) can be estimated from formulas (3.17) through (3.21) in Section 3.2.
1/ These methods are equally applicable to data based either on the date of last reporting for individual patients or on the common date.

2/ For simplicity, we assume for \( n = 1 \) and \( z = 1 \) for all \( x \), then \( c = 1 \) in the formula for \( \hat{c}_x \) [Chapter 4 (4.23)].

3/ To verify these computations find \( \hat{c}_0 \) with \( t = 1 \) using formula (2.19) of this chapter:

\[
\hat{c}_0 = \frac{1}{2} + \hat{p}_0 + \hat{p}_0 \hat{p}_1 + \ldots + \hat{p}_0 \hat{p}_1 \ldots \hat{p}_{y-1} + \hat{p}_{0y} \left\{ \frac{\hat{p}_t}{1 - \hat{p}_t} \right\}
\]

\[
= \frac{1}{2} + \left[ \hat{p}_{01} + \hat{p}_{02} + \ldots + \hat{p}_{0y} \right] + \hat{p}_{0y} \left\{ \frac{\hat{p}_{11}}{1 - \hat{p}_{11}} \right\}
\]

\[
= .5 + 6.02541 + .34277 (-0.9486) + 0.05706
\]

\[
= 12.8957 = 12.90
\]

This serves as a check of \( \hat{c}_0 = \frac{T_0}{T_0} = \frac{1,289,575}{100,000} = 12.89575 \) and thus of all \( I_x \) and \( T_x \).
Theoretical Justification of the Method of Life Table Construction in Chapter 3

Formulas (3.7) and (4.3) in Chapter 3, expressing the relation between the probability \( q_i \) and the corresponding death rate \( \mu_i \), were introduced as intuitive concepts for the purpose of application, but they can be derived from a theoretical viewpoint. Let \( \mu(x) \) be the force of mortality (mortality intensity function) at age \( x \). It is easy to see that the probability \( q_i \) that an individual alive at exact age \( x_i \) will die in interval \((x_i, x_i + n_i)\) is given by [cf., Appendix II, formula (2.7)]

\[
q_i = 1 - \exp\left\{-\int_{0}^{n_i} \mu(x_i + \xi) \, d\xi \right\}. \tag{1}
\]

For an individual at \( x_i \), let \( I_i \) be the number of deaths in \((x_i, x_i + 1)\).

Clearly, \( I_i = 1 \), if the individual dies in \((x_i, x_i + 1)\) with a probability \( q_i \) and \( I_i = 0 \), if the individual survives the interval, with a probability \( 1 - q_i \). Therefore, the expected number of deaths in \((x_i, x_i + 1)\) is \( E[I_i] = q_i \).

The mortality rate \( m_i \) is the ratio of the expected number of deaths \( q_i \) to the number of years an individual expects to live in the interval, or

\[
m_i = \frac{\frac{n_i}{1 - \exp\left\{-\int_{0}^{n_i} \mu(x_i + \xi) \, d\xi \right\}}}{\int_{0}^{n_i} \exp\left\{-\int_{0}^{y} \mu(x_i + \xi) \, d\xi \right\} dy}. \tag{2}
\]

Let a random variable \( \tau_i \) be the fraction of the interval \((x_i, x_i + n_i)\) lived by an individual who dies at an age included in the interval, so that \( \tau_i \) assumes values between 0 and 1. The expected value of \( \tau_i \) is the fraction of the last age interval of life, denoted by \( a_i \), i.e.,

\[
E(\tau_i) = a_i. \tag{3}
\]

\[
\tau_i = \frac{1 - \exp\left\{-\int_{0}^{n_i} \mu(x_i + \xi) \, d\xi \right\}}{\int_{0}^{n_i} \exp\left\{-\int_{0}^{y} \mu(x_i + \xi) \, d\xi \right\} dy}.
\]
For each time \( t, 0 \leq t \leq 1 \), the probability density function of \( \tau_1 \) is

\[
g(t)dt = \frac{n_1 \int_0^t \exp\{-\int_0^r \mu(x_1 + \xi) d\xi\} \mu(x_1 + n_1 t)n_1 \ dt}{q_i} \tag{4}
\]

\[0 \leq t \leq 1\]

The quantity on the right-hand side of (4) is the probability that an individual alive at \( x_1 \) will die in interval \((x_1 + n_1 t, x_1 + n_1 t + dt)\) providing that he dies in \((x_1, x_1 + n_1)\). According to the definition of \( \tau_1 \), this is also the probability that \( \tau_1 \) will assume values in \((t, t+dt)\), which is the density function \( g(t)dt \). The integral

\[
\int_0^{n_1 t} \exp\{-\int_0^r \mu(x_1 + \xi) d\xi\} \mu(x_1 + n_1 t)n_1 \ dt = 1 \tag{5}
\]

thus \( \tau_1 \) is a proper random variable. The expected value of \( \tau_1 \) may be computed as follows.

\[
a_i = E(\tau_1) = \int_0^1 t g(t) dt
\]

\[
= \int_0^{n_1 t} t \exp\{-\int_0^r \mu(x_1 + \xi) d\xi\} \mu(x_1 + n_1 t)n_1 dt \tag{6}
\]

Integrating the numerator in the last expression in (6) by parts gives the expression

\[
a_i = \frac{n_1 \int_0^{n_1 \mu(x_1 + \xi)d\xi} + \int_0^{\gamma} \exp\{-\int_0^{n_1 \mu(x_1 + \xi)d\xi}dy}{n_1 q_i} \tag{7}
\]

Substituting (1), (2), and (3) in the resulting formula (7) yields

\[
a_i = 1 - \frac{1}{q_i} + \frac{1}{n_1 n_i} \tag{8}
\]
Solving (8) for $q_1$, we obtain the fundamental relationship between $q_1$ and $m_1$

$$q_1 = \frac{n_1 m_1}{1 + (1-a_1)n_1 m_1}$$  \hspace{1cm} (9)

For age interval $(x, x+1)$ of one year ($n=1$), we write $a'_x$, $q_x$ and $m_x$ for $a_1$, $q_1$ and $m_1$, respectively, and have from (9)

$$q_x = \frac{m_x}{1 + (1-a'_x)m_x}$$  \hspace{1cm} (10)

where

$$q_x = 1 - \exp\left\{-\int_0^1 u(x+\xi)d\xi\right\} \quad \text{and} \quad m_x = \frac{q_x}{\int_0^t \exp\left\{-\int_0^1 u(x+\xi)d\xi\right\} dt}$$  \hspace{1cm} (11)

and

$$a'_x = \int_0^t \frac{1}{\mu(x+t)dt} \exp\left\{-\int_0^1 u(x+\xi)d\xi\right\}$$ \hspace{1cm} (12)

Formulas (9) and (10) are completely analogous to formulas (4.3) and (3.7) in Chapter 3.
APPENDIX II

STATISTICAL THEORY OF LIFE TABLE FUNCTIONS

1. Introduction

The concept of the life table originated in longevity studies of man, where it was always presented as a subject peculiar to public health, demography, and actuarial science. As a result, its development has not received sufficient attention in the field of statistics. Actually, the problems of mortality studies are similar to those of reliability theory and life testing, and they may be described in terms familiar to the statistically oriented mind. From a statistical point of view, human life is subject to chance. The life table systematically records the outcomes of many such experiments for a large number of individuals over a period of time. Thus the quantities in the table are random variables. Theoretical studies of the subject from a purely statistical point of view have been made; the probability distributions of life table functions have been devised and some optimum properties of these functions when they are used as estimates of the corresponding unknown quantities have been explored. The reader may refer to [Chiang, (1968). Chapter 10] for detail. Estimation problems concerning life table functions have been discussed by Grenander (1965). The purpose of this Appendix is to give a brief presentation of the theoretical aspects of the life table. A typical abridged life table is reproduced below.
Table 1
Life Table

<table>
<thead>
<tr>
<th>Age interval living (in years) at age $x_i$</th>
<th>Number living $l_i$</th>
<th>Proportion dying in interval $(x_i, x_{i+1})$</th>
<th>Fraction of last dying interval $(x_i, x_{i+1})$</th>
<th>Number dying in interval $(x_i, x_{i+1})$</th>
<th>Number of years lived in interval $(x_i, x_{i+1})$</th>
<th>Total number of years lived beyond age $x_i$ at age $x_i$</th>
<th>Observed expectation of life beyond age $x_i$ at age $x_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$ to $x_{i+1}$</td>
<td>$l_i$</td>
<td>$\hat{q}_i$</td>
<td>$a_i$</td>
<td>$d_i$</td>
<td>$L_i$</td>
<td>$T_i$</td>
<td>$\hat{e}_i$</td>
</tr>
<tr>
<td>$x_0$ to $x_1$</td>
<td>$l_0$</td>
<td>$\hat{q}_0$</td>
<td>$a_0$</td>
<td>$d_0$</td>
<td>$L_0$</td>
<td>$T_0$</td>
<td>$\hat{e}_0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_w$ and over</td>
<td>$l_w$</td>
<td>$\hat{q}_w$</td>
<td>$d_w$</td>
<td>$l_w$</td>
<td>$T_w$</td>
<td>$\hat{e}_w$</td>
<td></td>
</tr>
</tbody>
</table>

The following symbols are also used in the text:

$$p_{ij} = \Pr\{\text{an individual alive at age } x_i \text{ will survive to age } x_j\},$$

$$i \leq j; i, j=0,1,\ldots, \quad \text{(1.1)}$$

and

$$1 - p_{ij} = \Pr\{\text{an individual alive at age } x_i \text{ will die before age } x_j\},$$

$$i \leq j; i, j=0,1,\ldots. \quad \text{(1.2)}$$

When $x_j = x_{i+1}$, we drop the second subscript and write $p_i$ for $P_{i,i+1}$. No particular symbol is introduced for the probability $1 - p_{ij}$ except when $x_j = x_{i+1}$, in which case we let $1 - p_i = q_i$. 
Finally, the symbol $e_i$ is used to denote the true, unknown expectation of life at age $x_i$, estimated by the "observed expectation of life," $\hat{e}_i$.

All the quantities in the life table, with the exception of $l_0$ and $a_i$, are treated as random variables in this chapter. The radix $l_0$ is conventionally set equal to a convenient number, such as $l_0 = 100,000$, so that the value of $l_i$ clearly indicates the proportion of survivors to age $x_i$. We adopt the convention and consider $l_0$ a constant in deriving the probability distributions of other life table functions. The distributions of the quantities in columns $L_i$ and $T_i$ are not discussed because of their limited use. One remark should be made regarding the final age interval ($x$ and over): In a conventional table the last interval is usually an open interval, e.g., 95 and over; statistically speaking, $x$ is a random variable and is treated accordingly. However, discussion of this point, which is given in [Chiang, (1968), Chapter 10], will not be presented here. Throughout this appendix we shall assume a homogeneous population in which all individuals are subjected to the same force of mortality, and in which one individual's survival is independent of the survival of any other individual in the group.
2. Probability distribution of $\ell_x$, the number of survivors at age $x$.

The various functions of the life table are usually given for integral ages or for other discrete intervals. In the derivation of the distribution of survivors, however, age is more conveniently treated as a continuous variable with formulas derived for $\ell_x$, the number of individuals surviving the age interval $(0,x)$, for all possible values of $x$.

The probability distribution of $\ell_x$ depends on the force of mortality, or intensity of risk of death, $\mu(x)$, defined as follows:

$$u(x)\Delta + o(\Delta) = \Pr\{\text{an individual alive at age } x \text{ will die in interval } (x,x+\Delta)\}.$$  

Let the continuous random variable $X$ be the life span of a person so that the distribution function

$$F_X(x) = \Pr\{X \leq x\}$$  

is the probability that the individual will die prior to (or at) age $x$.

Consider now the interval $(0,x+\Delta)$ and the corresponding distribution function

$$F_X(x+\Delta) = \Pr\{X \leq x+\Delta\}.$$  

For an individual to die prior to $x+\Delta$ he must die prior to $x$ or else he must survive to $x$ and die during the interval $(x,x+\Delta)$. Therefore, the corresponding probabilities have the relation

$$F_X(x+\Delta) = F_X(x) + [1 - F_X(x)] [\mu(x)\Delta + o(\Delta)]$$  

or

$$\frac{F_X(x+\Delta) - F_X(x)}{\Delta} = [1 - F_X(x)] \left[ \mu(x) + \frac{o(\Delta)}{\Delta} \right].$$

Taking the limits of both sides of (2.4) as $\Delta \to 0$, we have the differential equation
\[ \frac{d}{dt} F_X(x) = \left[1 - F_X(x)\right] u(x) \]  

(2.5)

with the initial condition

\[ F_X(0) = 0. \]  

(2.6)

Integrating (2.5) and using (2.6) yields the solution

\[ 1 - F_X(x) = e^{-\int_0^x \mu(t) dt} = p_{0x} \]  

(2.7)

Equation (2.7) gives the probability that one individual alive at age 0 will survive to age \( x \). If there are \( l_0 \) individuals alive at age 0 who are subject to the same force of mortality, the number \( l_x \) of survivors at age \( x \) is clearly a binomial random variable with the probability \( p_{0x} \) of surviving to \( x \) and the probability distribution given by

\[ \Pr\{ l_x = k \} = \frac{l_0!}{k!(l_0-k)!} p_{0x}^k (1-p_{0x})^{l_0-k}, \quad k = 0, 1, \ldots, l_0. \]  

(2.8)

For \( x = x_i \), the probability that an individual will survive the age interval \( (0, x_i) \) is

\[ p_{0i} = \exp\left\{-\int_0^{x_i} \mu(t) dt\right\} \]  

(2.9)

and the probability distribution of the number of survivors, \( l_i \), is

\[ \Pr\{ l_i = k_i | l_0 \} = \binom{l_0}{k_i} p_{0i}^{k_i} (1-p_{0i})^{l_0-k_i}, \quad k_i = 0, 1, \ldots, l_0. \]  

(2.10)

The expected value and variance of \( l_i \) given \( l_0 \) are

\[ E(l_i | l_0) = l_0 p_{0i} \]  

(2.11)

and

\[ \sigma_{l_i}^2 | l_0 = l_0 p_{0i} (1-p_{0i}) \],  

(2.12)

respectively.
In general, the probability of surviving the age interval \((x_i, x_j)\) is

\[
p_{ij} = \exp \left\{ - \int_{x_i}^{x_j} \mu(t) \, dt \right\}, \text{ for } i \leq j
\]

with the obvious relation

\[
p_{aj} = p_{ai} p_{ij}, \quad \text{for } a \leq i \leq j.
\]

If we start with \(\ell_i\) individuals at \(x_i\), the number of survivors \(\ell_j\) at \(x_j\), for \(i \leq j\), is also a binomial random variable with the probability \(p_{ij}\) and

\[
\Pr(\ell_j = k_j | \ell_i) = \frac{\ell_i!}{k_j!(\ell_i-k_j)!} p_{ij}^k (1-p_{ij})^{\ell_i-k_j}, \quad k_j = 0, 1, \ldots, \ell_i
\]

with the expected value and variance given by

\[
E(\ell_j | \ell_i) = \ell_i p_{ij}
\]

and

\[
\frac{\sigma^2_j}{\ell_i} = \ell_i p_{ij} (1-p_{ij}).
\]

When \(j = i+1\), (2.15) becomes

\[
\Pr(\ell_{i+1} = k_{i+1} | \ell_i) = \frac{\ell_i!}{k_{i+1}!(\ell_i-k_{i+1})!} p_{i}^k (1-p_{i})^{\ell_i-k_{i+1}}.
\]

It is intuitively clear that given \(\ell_i\) people alive at age \(x_i\), the probability distribution of the number of people alive at \(x_j\), for \(x_j > x_i\), is independent of \(\ell_0, \ell_1, \ldots, \ell_{i-1}\). This means that for each \(k_j\)
Consequently,

\[ \Pr(l_j = k_j | l_0, l_1, \ldots, l_i) = \Pr(l_j = k_j) \]

(2.19)

Consequently,

\[ E(l_j | l_0, \ldots, l_i) = E(l_j | l_i) \]

and

\[ \sigma^2_{l_j | l_0, \ldots, l_i} = \sigma^2_{l_j | l_i} \]

In other words, for each \( u \) the sequence \( l_0, l_1, \ldots, l_u \) is a Markov process.

2.1. Mortality laws.

The survival probability in (2.7) has been known to life-table students for more than two hundred years. Unfortunately, it has not been given due recognition by investigators in statistics although differing forms of this function have appeared in various areas of research. We shall mention a few below in terms of the probability density function of \( X \),

\[ f_X(x) = \frac{dF_X(x)}{dx} = \mu(x)e^{-\int_0^x \mu(t)dt} \]

(2.20)

\[ = 0 \quad x < 0. \]

(i) Gompertz Distribution. In a celebrated paper on the law of human mortality, Benjamin Gompertz [1825] attributed death to two causes: chance, or the deterioration of the power to withstand destruction. In deriving his law of mortality, however, he considered only deterioration and assumed that man's power to resist death decreases at a rate proportional to the power itself. Since the force of mortality \( \mu(t) \) is a measure of man's susceptibility to death, Gompertz used the reciprocal \( 1/\mu(t) \) as a measure of man's resistance to death and thus arrived at the formula
\[
\frac{d}{dt} \left( \frac{1}{\mu(t)} \right) = -h \frac{1}{\mu(t)}, \tag{2.21}
\]

where \( h \) is a positive constant. Integrating (2.21) gives

\[
\ln\left( \frac{1}{\mu(t)} \right) = -ht + k \tag{2.22}
\]

which when rearranged becomes the Gompertz law of mortality

\[
\mu(t) = B e^t. \tag{2.23}
\]

The corresponding density function and distributions are given, respectively,

\[
f(x) = B c^x e^{-B[c^x - 1]/\ln c} \tag{2.24}
\]

and

\[
F_X(x) = 1 - \exp\left(-\frac{B}{\ln c} (c^x - 1)\right). \tag{2.25}
\]

(ii) Makeham's distribution. In 1860 W. M. Makeham suggested the modification

\[
\mu(t) = A + B c^t \tag{2.26}
\]

which is a restoration of the missing component, "chance" to the Gompertz formula. In this case, we have

\[
f(x) = [A + B c^x] \exp\left(-[A x + B (c^x - 1)/\ln c]\right) \tag{2.27}
\]

and

\[
F_X(x) = 1 - \exp\left(-[A x + B (c^x - 1)/\ln c]\right). \tag{2.28}
\]

(iii) Weibull distribution. When the force of mortality (or failure rate) is assumed to be a power function of \( t \), \( \mu(t) = \mu a t^{a-1} \), we have

\[
f(x) = \mu a x^{a-1} e^{-\mu x^a} \tag{2.29}
\]

and

\[
F_X(x) = 1 - e^{-\mu x^a} \tag{2.30}
\]
This distribution, recommended by W. Weibull [1939] for studies of the life span of materials, is used extensively in reliability theory.

(iv) **Exponential distribution.** If \( \mu(t) = \mu \) is a constant, then

\[
f(x) = \mu e^{-\mu x}
\]

and

\[
F_X(x) = 1 - e^{-\mu x}
\]

are formulas that play a central role in the problem of life testing (Epstein and Sobel [1953]).
3. Joint Probability Distribution of the Number of Survivors

Let us consider, for a given \( u \), the joint probability distribution of \( \ell_1, \ell_2, \ldots, \ell_u \) given \( \ell_0 \),

\[
\Pr(\ell_1 = k_1, \ldots, \ell_u = k_u | \ell_0).
\]

(3.1)

It follows from the Markovian property in (2.19) that

\[
\Pr(\ell_1 = k_1, \ell_2 = k_2, \ldots, \ell_u = k_u | \ell_0) = \Pr(\ell_1 = k_1 | \ell_0) \Pr(\ell_2 = k_2 | k_1) \ldots \Pr(\ell_u = k_u | k_{u-1}).
\]

(3.2)

Substituting (2.15) in (3.2) yields a chain of binomial distributions:

\[
\Pr(\ell_1 = k_1, \ell_2 = k_2, \ldots, \ell_u = k_u | \ell_0) = \frac{u!}{k_1!} \frac{k_1!}{k_1!} \frac{k_{i+1}!}{(k_i-k_{i+1})!} p_i^{k_i} (1-p_i)^{k_i-k_{i+1}}
\]

\[
k_{i+1} = 0, 1, \ldots, k_i, \quad \text{with } k_0 = \ell_0.
\]

(3.3)

Formula (3.3) shows that when a cohort of people is observed at regular points in time, the number of survivors, \( \ell_{i+1} \) to the end of the interval \((x_i, x_{i+1})\) has a binomial distribution depending solely on the number of individuals alive at the beginning of the interval \( \ell_i = k_i \).

The covariance between \( \ell_i \) and \( \ell_j \) may be obtained directly from (3.3); a somewhat simpler approach is the following. By definition

\[
\sigma_{\ell_i, \ell_j} = \text{Cov}(\ell_i, \ell_j) = \text{E}(\ell_i \ell_j) - \text{E}(\ell_i) \text{E}(\ell_j) = (\ell_0 p_{0i})(\ell_0 p_{0j})
\]

(3.4)

where

\[
\text{E}(\ell_i \ell_j) = \text{E}[\ell_i \text{E}(\ell_j | \ell_i)] = \text{E}[\ell_i \ell_j] = \text{E}[\ell_i^2] p_{ij}.
\]

(3.5)

Since \( \ell_i \) is a binomial random variable,

\[
\text{E}[\ell_i^2] = \ell_0 p_{0i} (1-p_{0i}) + [\ell_0 p_{0i}]^2.
\]

(3.6)
Substituting (3.5) and (3.6) successively in (3.4) and using the relationship $p_{01}p_{i} = p_{0j}$, we have the formula for the covariance

$$
\sigma_{\xi_i, \xi_j} = \xi_0 p_{0j} (1 - p_{0i}), \quad i \leq j; i, j = 0, 1, \ldots, u.
$$

(3.7)

When $j = i$, (3.7) reduces to the variance of $\xi_i$ (equation (2.12)). The correlation coefficient $\rho_{\xi_i, \xi_j}$ between $\xi_i$ and $\xi_j$, therefore, is given by

$$
\rho_{\xi_i, \xi_j} = \frac{p_{0i} (1 - p_{0i})}{\sqrt{p_{0i} (1 - p_{0i}) p_{0j} (1 - p_{0j})}} = \frac{p_{0i} (1 - p_{0i})}{p_{0i} (1 - p_{0j})},
$$

(3.8)

which is always positive whatever may be $0 < i < j$. For a given $i$, the correlation coefficient decreases as $x_j$ increases. This means that the larger the number of individuals alive at $x_i$, the more survivors there are likely to be at $x_j$; but the effect of the former on the latter decreases when $x_j$ becomes farther away from $x_i$. These results show that for a given $u$, $\xi_1, \ldots, \xi_u$ in the life table form a chain of binomial distributions; the joint probability distribution, the expected values, covariances and correlation coefficients are given in (3.3), (2.11), (3.7), and (3.8), respectively.
4. Joint Probability Distribution of the Numbers of Deaths

In a life table covering the entire life span of each individual in a given population, the sum of the deaths at all ages is equal to the size of the original cohort. Symbolically,

\[ d_0 + d_1 + \ldots + d_w = \gamma_0, \]  

(4.1)

where \( d_w \) is the number of deaths in the age interval \((x_w \text{ and over})\). Each individual in the original cohort has a probability \( p_{0i}q_i \) of dying in the interval \((x_i, x_{i+1})\), \( i=0,1,\ldots,w \). Since an individual dies once and only once in the span covered by the life table,

\[ p_{00}q_0 + \ldots + p_{0w}q_w = 1, \]  

(4.2)

where \( p_{00} = 1 \) and \( q_w = 1 \). Thus we have the well-known results: The numbers of deaths, \( d_0, \ldots, d_w \), in a life table have a multinomial distribution with the joint probability distribution

\[ \Pr\{d_0 = \delta_0, \ldots, d_w = \delta_w\} = \frac{l_0!}{\delta_0! \ldots \delta_w!} (p_{00}q_0)^{\delta_0} \ldots (p_{0w}q_w)^{\delta_w}; \]  

(4.3)

the expectation, variance, and covariance are given, respectively, by

\[ E(d_1|\gamma_0) = l_0p_{0i}q_i, \]  

(4.4)

\[ \sigma^2 = l_0p_{0i}q_i(1-p_{0i}q_i), \]  

(4.5)

and

\[ \sigma_{d_i,d_j} = -l_0p_{0i}q_ip_{0j}q_j \quad \text{for } i \neq j; i,j=0,1,\ldots,w. \]  

(4.6)
In the discussion above, age 0 was chosen only for simplicity. For any given age, say \( x_a \), the probability that an individual alive at age \( x_a \) will die in the interval \((x_i, x_{i+1})\) subsequent to \( x_a \) is \( p_{ai}q_i \) and the sum

\[
\sum_{i=a}^w p_{ai}q_i = 1,
\]  

(4.7)

and thus the numbers of deaths in intervals beyond \( x_a \) also have a multinomial distribution.
5. Optimum Properties of \( \hat{p}_j \) and \( \hat{q}_j \)

The quantity \( \hat{q}_j \) (or \( \hat{p}_j \)) is an estimator of the probability that an individual alive at age \( x_j \) will die in (or survive) the interval \((x_j, x_{j+1})\), with

\[
\hat{p}_j + \hat{q}_j = 1, \quad j = 0, l, \ldots.
\]  

(5.1)

Therefore, \( \hat{p}_j \) and \( \hat{q}_j \) have the same optimum properties. For convenience, we consider \( \hat{p}_j \) in the following discussion.

5.1. Maximum likelihood estimator of \( \hat{p}_j \). The joint probability distribution (3.3), when expressed in terms of the random variables \( \xi_1, \ldots, \xi_u \), may be rewritten as

\[
L = \prod_{i=0}^{u-1} \frac{\xi_i!}{(\xi_i - \xi_{i+1})!} p_i^{\xi_{i+1}} (1-p_i)^{\xi_i - \xi_{i+1}}
\]  

(5.2)

which is known as the likelihood function of \( \xi_1, \ldots, \xi_u \). When the right hand side of (5.2) is maximized with respect to \( p_j \), we have the maximum likelihood estimators, say \( \hat{p}_j \). In this case, the maximizing values \( \hat{p}_j \) can be derived by differentiation. Letting

\[
\log L = C + \sum_{i=0}^{u-1} \xi_{i+1} \log p_i + \sum_{i=0}^{u-1} (\xi_i - \xi_{i+1}) \log(1-p_i)
\]  

(5.3)

setting the derivative equal to zero,

\[
\frac{\partial}{\partial p_j} \log L = \frac{\xi_{i+1}}{p_j} - \frac{\xi_i - \xi_{i+1}}{1-p_j} = 0,
\]  

(5.4)

and solving the equations (5.4), we have the maximum likelihood estimators

\[
\hat{p}_j = \frac{\xi_{i+1}}{\xi_j} \quad j = 0,1,\ldots,u-1.
\]  

(5.5)
It should be noted that if for some age \( x^w \) all the \( i \) individuals alive at \( x^w \) die within the interval \((x^w, x^w + 1)\), then \( i = 0 \) for all \( i > w \), so that there is no contribution to the likelihood function beyond the \( w \)th factor. Consequently, the maximum-likelihood estimator in (5.5) is defined only for \( \hat{\lambda}_j > 0 \). With this understanding, let us compute the first two moments.

We have shown in Section 2 that, given \( \lambda_j > 0 \), the number \( \lambda_j + 1 \) has the binomial distribution, therefore

\[
E[\hat{\lambda}_j] = E\left(\frac{\lambda_j + 1}{\lambda_j}\right) = E\left[\frac{1}{\lambda_j} E(\lambda_j + 1 | \lambda_j)\right] = p_j, \tag{5.6}
\]

and \( \hat{\lambda}_j \) and hence \( \hat{q}_j \) are unbiased estimators of the corresponding probabilities. Direct computation gives also

\[
E[\hat{\lambda}_j^2] = E\left(\frac{1}{\lambda_j}\right) p_j (1 - p_j) + p_j^2 \tag{5.7}
\]

and the variance

\[
\sigma_{\hat{\lambda}_j}^2 = E\left(\frac{1}{\lambda_j}\right) p_j (1 - p_j) = \sigma_{\hat{q}_j}^2. \tag{5.8}
\]

When \( \lambda_0 \) is large, (5.8) may be approximated by

\[
\sigma_{\hat{\lambda}_j}^2 = \frac{1}{E(\lambda_j)} p_j (1 - p_j). \tag{5.9}
\]

Justification of (5.9) is left to the reader.

For the covariance between \( \hat{\lambda}_j \) and \( \hat{\lambda}_k \) for \( j < k \), we require that \( \lambda_k \) and hence \( \lambda_j \) and \( \lambda_j + 1 \) be positive and compute the conditional expectation
from which it follows that

$$E[\hat{p}_k | \hat{p}_j] = E[\frac{\ell_{k+1}}{\ell_k} | \hat{p}_j] = E[\frac{1}{\ell_k} E(\ell_{k+1} | \hat{p}_j)] \hat{p}_j = \hat{p}_k = E(\hat{p}_k), \quad (5.10)$$

and that

$$E[\hat{p}_j \hat{p}_k] = E[\hat{p}_j E(\hat{p}_k | \hat{p}_j)] = E[\hat{p}_j] E[\hat{p}_k]$$

Observe that formula (5.11) of zero covariance holds only for proportions in two non-overlapping age intervals. If the two intervals considered both begin with age $x_a$ but extend to ages $x_j$ and $x_k$, respectively, the covariance between the proportions $\hat{p}_{aj}$ and $\hat{p}_{ak}$ is not equal to zero. Easy computation shows that

$$\sigma_{\hat{p}_{aj}, \hat{p}_{ak}} = 0. \quad (5.11)$$

When $k = j$, (5.12) becomes the variance of $\hat{p}_{aj}$.

Although $\hat{p}_j$ and $\hat{p}_k$ have zero covariance, they are not independently distributed. For example

$$\Pr(\hat{p}_1 = .5 | \hat{p}_0 = 1) \neq \Pr(\hat{p}_1 = .5 | \hat{p}_0 = .8).$$

Thus we have shown that the quantities $\hat{p}_j$ and $\hat{q}_j$ in the life table are the unbiased, maximum-likelihood estimators of the corresponding probabilities $p_j$ and $q_j$. 
6. Distribution of $\hat{e}_a$, the Observed Expectation of Life at Age $x_a$

The observed expectation of life summarizes the mortality experience of a population from a given age to the end of the life span. At age $x_i$, the expectation expresses the average number of years remaining to each individual living at that age if all individuals are subjected to the estimated probabilities of death $\hat{q}_j$ for $j \geq i$. This is certainly the most useful column in the life table.

To avoid confusion in notation, let $\alpha$ denote a fixed number and $x_\alpha$ a particular age. We are interested in the distribution of $\hat{e}_a$, the observed expectation of life at age $x_a$. Consider $\hat{e}_a$, the number of survivors to age $x_a$, and let $Y_\alpha$ denote the future lifetime beyond age $x_a$ of a particular individual. Clearly, $Y_\alpha$ is a continuous random variable that can take on any non-negative real value. Let $y_\alpha$ be the value that the random variable $Y_\alpha$ assumes, then $x_\alpha + y_\alpha$ is the entire life span of the individual. Let $f(y_\alpha)$ be the probability density function of the random variable $Y_\alpha$, and let $dy_\alpha$ be an infinitesimal time interval. Since $Y_\alpha$ can assume values between $y_\alpha$ and $y_\alpha + dy_\alpha$, if and only if the individual survives the age interval $(x_\alpha, x_\alpha + y_\alpha)$ and dies in the interval $(x_\alpha + y_\alpha, x_\alpha + y_\alpha + dy_\alpha)$, we have

$$f(y_\alpha)dy_\alpha = e^{\int_{x_\alpha}^{x_\alpha + y_\alpha} \mu(t)dt} \cdot \mu(x_\alpha + y_\alpha)dy_\alpha, \quad y_\alpha > 0.$$  \hspace{1cm} (6.1)

Function $f(y_\alpha)$ in (6.1) is a proper probability density function since it is never negative and since the integral of the function from $y_\alpha = 0$ to $y_\alpha = \infty$ is equal to unity. Clearly, $f(y_\alpha)$ can never be negative whatever the value of $y_\alpha$. To evaluate the integral
we define a quantity $\phi$

$$\phi = \int_{x_a}^{x_a+y_a} \mu(\tau) d\tau = \int_0^{y_a} \mu(x_a + t) dt$$  \hspace{1cm} (6.3)$$

and substitute the differential

$$d\phi = \mu(x_a + y_a) dy_a$$  \hspace{1cm} (6.4)$$

in the integral to give the solution

$$\int_0^\infty f(y_a) dy_a = \int_0^\infty e^{-\phi} d\phi = 1.$$  \hspace{1cm} (6.5)$$

The mathematical expectation of the random variable $Y_a$ is the expected length of life beyond age $x_a$, and thus is the true expectation of life at age $x_a$. In accordance with the definition given the symbol $e_a$, we may write

$$e_a = \int_0^\infty y_a f(y_a) dy_a = \int_{x_a}^{x_a+y_a} y_a e^{\int_{x_a}^{x_a+y_a} \mu(\tau) d\tau} dy_a.$$  \hspace{1cm} (6.6)$$

Thus the expectation $e_a$ and the variance

$$\sigma_{Y_a}^2 = \int_0^\infty (y-a)^2 f(y_a) dy_a$$  \hspace{1cm} (6.7)$$

both depend on the intensity of risk of death $\mu(\tau)$. 
The expectation of life at age $x$ is conventionally defined as

$$e_x = \int_0^\infty e^{-x} \mu(t) dt$$

(6.8)

It is instructive to prove that the two alternative definitions (6.6) and (6.8) are identical. Let $u = y_x$, $du = dy_x$,

$$v = -e^{-x} \mu(t) dt$$

(6.9)

and

$$dv = e^{-x} \mu(x+y_x) dy_x .$$

(6.10)

Integrating (6.6) by parts gives

$$\int_0^\infty - \int_x^{x+y_x} \mu(t) dt$$

$$e_x e^{x} \mu(x+y_x) dy_x$$

(6.11)

$$= -y_x e^{-x} \mu(t) dt \bigg|_0^\infty + \int_0^\infty e^{-x} \mu(t) dt$$
The first term on the right vanishes and the second term is the same as (6.8), proving the identity.

6.1. The variance of the expectation of life, $\hat{e}_a$. The future lifetimes of $\lambda_a$ survivors may be regarded as a sample of $\lambda_a$ independent and identically distributed random variables, $Y_{ak}$, $k=1, \ldots, \lambda_a$, each of which has the probability density function (6.1), the expectation (6.6), and the variance (6.7). According to the central limit theorem, for large $\lambda_a$ the distribution of the sample mean

$$\bar{Y}_a = \frac{1}{\lambda_a} \sum_{k=1}^{\lambda_a} Y_{ak}$$

(6.12)

is approximately normal with an expectation as given in (6.6) and a variance $\sigma_{Y_a}^2 / \lambda_a$. It has been shown in Chapter 4, Section 3, that the sample mean $\bar{Y}_a$ is equal to the observed expectation of life $\hat{e}_a$, or

$$\bar{Y}_a = \hat{e}_a$$

(6.13)

Therefore, the variance of $\hat{e}_a$ is also $\sigma_{Y_a}^2 / \lambda_a$. For practical purposes, we need to have a formula for the variance of $\hat{e}_a$ which can be estimated for the cohort and the current life tables. The formula of $\hat{e}_a$ is given by

$$\hat{e}_a = \frac{1}{\frac{1}{\lambda_a}} \left[ \sum_{i=1}^{\nu-1} \left( n_i (C_i - D_i) + \lambda_1 \frac{n_i d_{i+1}}{n_{i+1}} + \lambda \frac{n_i d_1}{n_1} \right) \right]$$

(6.14)
Using the relation \( d_i = l_i - l_{i+1} \), \( i = 0, \ldots, w-1 \), we rewrite (6.14) as

\[
\hat{e}_a = a_n a + \sum_{i=a+1}^{w} c_i \frac{t_i}{l_a} = a_n a + \sum_{i=a+1}^{w} c_i \hat{p}_a l
\]  

(6.15)

where \( c_i = (1-a_{i-1})n_{i-1} + a_i n_i \). Because the proportion \( \hat{p}_a \) in (6.15) is an unbiased estimate of \( p_a \), the expectation of \( \hat{e}_a \) as given by (6.6) is simply

\[
e_a = a_n a + \sum_{i=a+1}^{w} c_i p_a^i, \quad a=0,1,\ldots,w.
\]  

(6.16)

The observed expectation of life as given in (6.15) is a linear function of \( \hat{p}_a \), which in the current life table is computed from

\[
\hat{p}_a = \hat{p}_a \hat{p}_{a+1} \cdots \hat{p}_{j-1}, \quad j=a+1,\ldots,w.
\]  

(6.17)
Clearly, the derivatives taken at the true point \((p_0, p_{a+1}, \ldots, p_{j-1})\) are

\[
\frac{3}{\partial p_i} \hat{p}_a = p_{ai} p_{i+1,j}, \quad \text{for } a < i < j;
\]

\[
= 0, \quad \text{for } i \geq j.
\]

Hence

\[
\frac{3}{\partial p_i} \hat{e}_a = \frac{w}{j=i+1} c_j p_{ai} p_{i+1,j}
\]

\[
= p_{ai} \left[ c_{i+1} + \sum_{j=i+2}^{w} c_j p_{i+1,j} \right]
\]

\[
= p_{ai} \left[ e_{i+1} + (1-a_i)n_i \right].
\]

Because of (6.18), the derivative (6.19) vanishes when \(i = w\). Since it has been shown in Section 5 [cf. equation (5.11)] that the covariance between proportions of survivors of two non-overlapping age intervals is zero, the variance of the observed expectation of life may be computed from the following

\[
\sigma^2_{\hat{e}_a} = \sum_{i=a}^{w-1} \left\{ \frac{3}{\partial p_i} \hat{e}_a \right\}^2 \sigma^2_{\hat{p}_i}.
\]

Substitution of (6.19) in (6.20) gives the formula

\[
\sigma^2_{\hat{e}_a} = \sum_{i=a}^{w-1} p_{ai}^2 \sigma^2_{\hat{e}_{i+1} + (1-a_i)n_i}^2 \sigma^2_{q_i}, \quad a=0,1,\ldots,w-1. \quad (6.21)
\]
Thus we have

**Theorem:** If the distribution of deaths in the age interval \((x_i, x_{i+1})\) is such that, on the average, each of the \(d_i\) individuals lives \(a_i n_i\) years in the interval, for \(i = \alpha, \alpha+1, \ldots, \omega\), then as \(n\) approaches infinity, the probability distribution of the observed expectation of life at age \(x_\alpha\), as given by (6.15), is asymptotically normal and has the mean and variance as given by (6.16) and (6.21), respectively.
APPENDIX III

THE THEORY OF COMPETING RISKS

A Historical Note - Daniel Bernoulli's Work

The concept of competing risks is not new; it seems to have originated in a controversy over the value of vaccination. The first systematic discussion of the problem was by Daniel Bernoulli in 1760 in his article entitled, "Essai d'une nouvelle analyse de la mortalité causee par la petite verole et des avantages de l'inoculation pour la prevenir." The main objective of the memoir was to determine the mortality caused by smallpox at various ages. Since his work created much discussion in his time and opened up a new area of study in competing risks, it may be appropriate to review briefly Daniel Bernoulli's approach.

Let $y_x$ denote the number who survive to age $x$; among them $s_x$ have not had smallpox. Assume that in a year smallpox attacks one out of $n$ individuals who have not had the disease, and one out of every $m$ individuals who contract the disease dies. Both $n$ and $m$ are assumed to be constant. Within the time element $dx$, the number who die is $-d_{xx}$, and the number who die from smallpox is

$$s_x dx \over mn$$

and therefore the number who die from other causes is

$$-d_{xx} - s_x dx \over mn$$

Now the number of those who have not had smallpox will decrease during the time element $dx$ through contracting smallpox ($s_x dx/n$) and through death (a proportion $s_x/2$ of that in (2)). Denoting this reduction by $-ds_x$, we have the equation,

$$\frac{s}{2}$$
Equation (3) may be rearranged to yield

\[
- \frac{ds}{x} = \frac{s}{n} \frac{dx}{x} - \frac{x}{\ell} \left( \frac{d\ell}{x} + \frac{s}{mn} \frac{dx}{x} \right) .
\]  

or

\[
\frac{s}{x} \frac{d\ell}{x} - \frac{\ell}{s} \frac{ds}{x} = \left( m \frac{x}{s} - 1 \right) \frac{dx}{mn} .
\]  

Integrating both sides of (5) gives

\[
(m \frac{x}{s} - 1)^n = e^{x+c} .
\]  

or

\[
\frac{s}{x} = \frac{m\ell}{1+e^{(x+c)/n}} .
\]  

To determine the constant of integration c, we observe that at x=0, s_0 = \ell_0 so that e^{c/n} = (m-1), and thus

\[
\frac{s}{x} = \frac{m\ell}{1+(m-1)e^{x/n}} .
\]  

Using formula (7) and assuming m = n = 8, Bernoulli calculated \ell_x and s_x on the basis of Halley's table of Breslow.

Bernoulli also derived a formula for the number of survivors had there been no smallpox. Using a similar approach, he showed that this number of survivors, denoted by z_x, is given by
\[
Z_x = \frac{m_l \cdot e^{x/n}}{1 + (m-1)e^{x/n}}
\]  

The right-hand side of (8) increases as either \( m \) or \( n \) decreases and approaches the limit \( m \cdot l_x/(m-1) \) as \( x \) increases indefinitely.

After discussing the subject of the mortality from smallpox, Daniel Bernoulli proceeded to the discussion of inoculation. He admitted that there was some danger in inoculation against smallpox, but he found that on the whole it was advantageous. Based on his calculations, he concludes that inoculation would increase the average length of life by three years.

An important assumption in Daniel Bernoulli's solution of the problem was the constant incidence rate \( (1/n) \) and constant case fatality rate \( (1/m) \). D'Alembert (1717-1783), Trembley (1749-1811), and Laplace (1749-1827) all had considered the case when \( n \) and \( m \) both are functions of age \( x \). It was D'Alembert who was the most critical of Bernoulli's solution. Although he too recognized the value of inoculation, he felt that Bernoulli had overestimated its advantage. While he failed to provide a neat solution to the problem, D'Alembert brought up a significant distinction between the physical life and the real life of an individual. By the physical life, he meant life in the ordinary sense; by the real life he meant the portion of existence during which the individual enjoys life in a disease-free state. Theoretical pursuit of this aspect of the problem, however, was not in evidence either in D'Alembert's work or in that of others. A detailed account of the controversy may be found in Todhunter [1949]. Thus, Bernoulli, D'Alembert, Trembley and Laplace each derived a method of determining the change in population composition that would take place if smallpox were eliminated as a cause of death. It was Makeham [1874], however, who first formulated a theory of decremental forces and explored its practical applications.
Actuarial mathematicians have applied Makeham's work to develop multiple-decrement theory in the study of life contingencies. For a detailed account of the theory, reference should be made to C. W. Jordan [1952]. In the last thirty years, the theory of competing risks have attracted much attention in the field of health and statistics. Creville [1948] discussed deterministically multiple decrement tables. Fix and Neyman [1951] studied the problem of competing risks for cancer patients; and Chiang [1961a] approached the problem from a stochastic viewpoint. Other interesting studies include those by Berkson and Elveback [1960], Berman [1961], Cornfield [1957], and Kimball [1958], [1969], David [1970], Pike [1970], Mantel and Bailar [1970] and Chiang [1970].

I. Introduction

Although the basic characteristics of mortality studies are that death is not a repetitive event and that usually death is attributed to a single cause, in cause-specific mortality studies the various risks competing for the life of an individual must be considered as well. For example, in an investigation of congenital malformation as a cause of infant death, some subjects would die from other causes such as tuberculosis. These infants have no chance either of dying from congenital malformation or of surviving the first year of life. What then would be the contribution of their survival experience to such a mortality study and what adjustment would have to be made for the competing effect of tuberculosis in the assessment of congenital malformation as a cause of death? Competing risks must also be taken into account in studies of the relative susceptibility of individuals in different illness states to other diseases. For instance, would people suffering from arteriosclerotic heart disease be more likely to die from pneumonia than those without a heart condition? Any meaningful comparison between the two groups with respect to their susceptibility to pneumonia would have to evaluate the effect of arteriosclerosis as a competing risk. The following three types of probability of death from a specific cause are necessary for an understanding of the study of survival as well as the application of life table methodology to such problems as those above.
1. **Crude probability**: The probability of death from a specific cause in the presence of all other risks acting in a population, or

\[ Q_{i5} = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from cause } R_5 \text{ in the presence of all other risks in the population} \} \]

2. **Net probability**: The probability of death if a specific risk is the only risk in effect in the population or, conversely, the probability of death if a specific risk is eliminated from the population.

\[ q_{i5} = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ if } R_5 \text{ is the only risk acting on the population} \} \]

\[ q_{i5} = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ if } R_5 \text{ is eliminated as a risk of death} \} \]

3. **Partial crude probability**: The probability of death from a specific cause when another risk (or risks) is eliminated from the population.

\[ Q_{i5,1} = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from } R_5 \text{ if } R_1 \text{ is eliminated from the population} \} \]

\[ Q_{i5,12} = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from } R_5 \text{ if } R_1 \text{ and } R_2 \text{ are eliminated from the population} \} \]

When the cause of death is not specified, we have the probabilities

\[ p_i = \Pr\{ \text{an individual alive at } x_i \text{ will survive the interval } (x_i, x_{i+1}) \} \]

and

\[ q_i = \Pr\{ \text{an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \} \]

with \( p_i + q_i = 1 \).

The use of the terms "risk" and "cause" needs clarification. Both terms may refer to the same condition but are distinguished here by their position in time.
relative to the occurrence of death. Prior to death the condition referred to is a risk; after death the same condition is the cause. For example, tuberculosis is a risk of dying to which an individual is exposed, but tuberculosis is also the cause of death if it is the disease from which the individual eventually dies.

In the human population the net and partial crude probabilities cannot be estimated directly, but only through their relations with the crude probability. The study of these relations is part of the problem of "competing risks," or "multiple decrement." Formulas expressing relations between net and crude probabilities have been developed by assuming either a constant intensity of risk of death (force of mortality) or a uniform distribution of deaths. We will review these formulas in this Appendix assuming a constant relative intensity. Partial crude probabilities have not received due attention in view of their often indispensable role in studies of cause-specific mortality. Their relations with the corresponding crude probabilities will also be discussed.
2. Relations between Crude, Net and Partial Crude Probabilities

Suppose that \( r \) risks of death are acting simultaneously on each individual in a population, and let these risks be denoted by \( R_1, \ldots, R_r \). For each risk, \( R_\delta \), there is a corresponding intensity function (or force of mortality) \( \mu(t; \delta) \) such that,

\[
\mu(t; \delta) \Delta + o(\Delta) = \Pr\{\text{an individual alive at time } t \text{ will die in interval} \ (t, t+\Delta) \text{ from risk } R_\delta \}, \ \delta = 1, \ldots, r,
\]

and the sum

\[
\mu(t; 1) + \cdots + \mu(t; r) = \mu(t)
\]

is the total intensity (or the total force of mortality). Although for each risk \( R_\delta \) the intensity \( \mu(t; \delta) \) is a function of time \( t \), we assume that within the time interval \( (x_i, x_{i+1}) \) the ratio

\[
\frac{\mu(t; \delta)}{\mu(t)} = c_{i\delta}
\]

is independent of time \( t \), but is a function of the interval \( (x_i, x_{i+1}) \) and risk \( R_\delta \). Assumption (2.3), which is known as the proportionality assumption, permits the risk-specific intensity \( \mu(t; \delta) \) to vary in absolute magnitude, but requires that it remain a constant proportion of the total intensity in an interval.

Consider death without specification of cause. The probability that an individual alive at \( x_i \) will survive the interval \( (x_i, x_{i+1}) \) is

\[
p_i = e^{-\int_{x_i}^{x_{i+1}} \mu(t) \ dt}, \quad i = 0, 1, \ldots
\]

and the probability of his dying in the interval is \( q_i = 1 - p_i \) (see formula (2.9) in Appendix II).
To derive the crude probability of dying from risk $R_\delta$, we consider a point $t$ within the interval $(x_i, x_{i+1})$. The probability that an individual alive at $x_i$ will die from $R_\delta$ in interval $(t, t+dt)$ is

$$
Q_{i\delta} = \int_{x_i}^{x_{i+1}} e^{-\int \mu(\tau) d\tau} \nu(t_\delta) dt
$$

(2.5)

where the exponential function is the probability of surviving from $x_i$ to $t$ when all risks are acting, and the factor $\nu(t_\delta) dt$ is the instantaneous death probability from risk $R_\delta$ in time interval $(t, t+dt)$. Summing (2.5) over all possible values of $t$, for $x_i \leq t < x_{i+1}$, gives the crude probability

$$
Q_{i\delta} = \int_{x_i}^{x_{i+1}} \nu(t_\delta) dt \cdot e^{-\int \mu(\tau) d\tau}
$$

(2.6)

Under the proportionality assumption (2.3), (2.6) may be rewritten as

$$
Q_{i\delta} = \frac{\nu(t_\delta)}{\nu(t)} \int_{x_i}^{x_{i+1}} e^{-\int \mu(\tau) d\tau} \nu(t) dt.
$$

(2.7)

Integrating gives

$$\nu(t; \delta) = \frac{\nu(t_\delta)}{\nu(t)} \left[ \int_{x_i}^{x_{i+1}} \nu(t) dt \right] \cdot \left( 1 - e^{-\int \mu(\tau) d\tau} \right) = \nu(t_\delta) \nu(t) q_i ;
$$

(2.8)

hence

$$\frac{\nu(t; \delta)}{\nu(t)} = \frac{Q_{i\delta}}{q_i}, \quad x_i \leq t < x_{i+1}; \quad \delta = 1, \ldots, r.
$$

(2.9)
Equation (2.9) is obvious, for if the ratio of the risk-specific intensity to the total intensity is constant throughout an interval, this constant should also be equal to the ratio of the corresponding probabilities of dying over the entire interval. Equations (2.2) and (2.9) imply a trivial equality

\[ Q_{i1} + \cdots + Q_{i,r} = q_i, \quad i=0,1,\ldots. \]  

**Remark 1.** Equation (2.9) suggests a similarity between the intensity functions \( \mu(t;\delta) \) and the probability \( \nu(t;\delta) \). For example, from (2.9) we have

\[ \frac{\mu(t;\delta)}{\mu(t;c)} = \frac{Q_{i,\delta}}{Q_{i,c}}, \]

so that the relative magnitude between any two probabilities is equal to the relative magnitude between the corresponding intensity functions. However, when several sets of values are considered, the variation among \( Q_{i,\delta} \) may be quite different from the variation among \( \mu(t;\delta) \). To illustrate, let \( \mu(t;\delta) = \mu(i;\delta) \) for \( x_i \leq t \leq x_{i+1} \) and \( \delta = 1,\ldots,r \); so that \( \mu(t) = \mu(i) \). Then (2.9) implies that

\[ \nu(i;\delta) = -\frac{Q_{i,\delta}}{q_i} \ln (1-q_i). \]  

Suppose we let \( Q_{i1} \) increase but keep \( Q_{i2},\ldots,Q_{ir} \) unchanged. The intensity functions \( \mu(i;2),\ldots,\mu(i;r) \) will not remain constant, but rather they increase with the increasing values of \( Q_{i1} \) (or with increasing values of \( q_1 \), since \( q_1 = Q_{i1}+\cdots+Q_{ir} \)). In other words, the function

\[ h(q_1) = -\frac{1}{q_1} \ln (1-q_1) \]

on the right-hand side of (2.12) is a monotonically increasing function of \( q_1 \). Taking the derivative of \( h(q_1) \) with respect to \( q_1 \) yields
\[ h'(q_1) = \frac{1}{q_1} \left[ \ln(1-q_1) + \frac{q_1}{1-q_1} \right] \]

(2.13)

\[ = \sum_{n=2}^{\infty} \frac{n-1}{n} q_1^{n-2} \]

since \( q_1 \) is between 0 and 1. The last expression in (2.13) is always positive for positive values of \( q_1 \). Hence the function \( h(q_1) \) increases with \( q_1 \) and \( \mu(i; \delta) \) increases with \( Q_{i1} \), as required to be shown.

A numerical example for \( r = 3 \) risks is given below. It is also easy to see that

\[ \frac{Q_{i1}}{\mu(i,1)} = \frac{Q_{i2}}{\mu(i,2)} = \frac{Q_{i3}}{\mu(i,3)} \]

(2.14)

### Table 1. Probabilities and Intensity Functions

<table>
<thead>
<tr>
<th>( Q_{i1} )</th>
<th>( Q_{i2} )</th>
<th>( Q_{i3} )</th>
<th>( q_i )</th>
<th>( \mu(i;1) )</th>
<th>( \mu(i;2) )</th>
<th>( \mu(i;3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.01</td>
<td>.01</td>
<td>.30</td>
<td>.32</td>
<td>.0121</td>
<td>.0121</td>
<td>.3615</td>
</tr>
<tr>
<td>.05</td>
<td>.01</td>
<td>.30</td>
<td>.36</td>
<td>.0620</td>
<td>.0124</td>
<td>.3719</td>
</tr>
<tr>
<td>.10</td>
<td>.01</td>
<td>.30</td>
<td>.41</td>
<td>.1287</td>
<td>.0129</td>
<td>.3860</td>
</tr>
<tr>
<td>.25</td>
<td>.01</td>
<td>.30</td>
<td>.56</td>
<td>.3665</td>
<td>.0147</td>
<td>.4398</td>
</tr>
<tr>
<td>.50</td>
<td>.01</td>
<td>.30</td>
<td>.81</td>
<td>1.0251</td>
<td>.0205</td>
<td>.6151</td>
</tr>
</tbody>
</table>

2.1. Relations between crude and net probabilities.

The net probability of death in the interval \((x_i, x_{i+1})\) when \( R_\delta \) is the only operating risk is obviously

\[ q_{i\delta} = 1 - e^{-\int_{x_i}^{x_{i+1}} \mu(t; \delta) dt} \]

(2.15)
which, in view of (2.3), can be written as

\[
q_{i\delta} = 1 - e^{-\frac{\mu(t;\delta)}{\mu(t)} \int_{x_i}^{x_{i+1}} \mu(t) \, dt} = 1 - p_i \frac{\mu(t;\delta)}{\mu(t)} .
\]  

(2.16)

With equation (2.9), (2.16) gives the relation between the net and the crude probabilities,

\[
q_{i\delta} = 1 - p_i \frac{Q_{i\delta}}{q_i}, \quad \delta = 1, \ldots, r.
\]  

(2.17)

Formula (2.17) may be simplified. Using the \( p_i = 1 - q_i \), the second term on the right-hand side is expanded in terms of Newton's binomial series,

\[
p_i \frac{Q_{i\delta}}{q_i} = (1 - q_i)^{Q_{i\delta}/q_i} = 1 + \frac{Q_{i\delta}}{q_i} (-q_i) + \cdots + \binom{Q_{i\delta}/q_i}{k} (-q_i)^k + \cdots
\]  

(2.18)

where the binomial coefficient is defined as follows:

\[
\binom{Q_{i\delta}/q_i}{k} = \frac{1}{k!} \frac{Q_{i\delta}}{q_i} \left( \frac{Q_{i\delta}}{q_i} - 1 \right) \cdots \left( \frac{Q_{i\delta}}{q_i} - k + 1 \right)
\]  

for \( k = 0, 1, 2, \ldots \).

Because of small values of \( q_i \), the first four terms of the infinite series in (2.18) give a good approximation. As a result, we have

\[
p_i \frac{Q_{i\delta}}{q_i} = 1 - Q_{i\delta} - \frac{1}{2} Q_{i\delta} (q_i - Q_{i\delta}) - \frac{1}{6} Q_{i\delta} (q_i - Q_{i\delta}) (2q_i - Q_{i\delta}) .
\]  

(2.20)

Substituting (2.20) in (2.17) yields the relationship

\[
q_{i\delta} = Q_{i\delta} \left[ 1 + \frac{1}{2} (q_i - Q_{i\delta}) + \frac{1}{6} (q_i - Q_{i\delta}) (2q_i - Q_{i\delta}) \right].
\]  

(2.21)

When the first three terms of the binomial series are taken, we have

\[
q_{i\delta} = Q_{i\delta} \left[ 1 + \frac{1}{2} (q_i - Q_{i\delta}) \right],
\]  

(2.21a)

which may be used when \( q_i \) is extremely small.
The net probability of death when risk $R_\delta$ is eliminated can be derived in a similar way. When $R_\delta$ is eliminated as a cause of death, the force of mortality is $\mu(t) - \mu(t;\delta)$. In this case, the probability that an individual alive at $x_i$ will die in $(t, t+dt)$ is

$$ e^{\int_{x_i}^{x_{i+1}} [\mu(t) - \mu(t;\delta)] dt} $$

where the exponential function is the probability of his surviving from $x_i$ to $t$ and $[\mu(t) - \mu(t;\delta)] dt$ is the probability that he will die in the time element $(t, t+dt)$. For different values of $t$, the corresponding events associated with the probability in (2.22) are mutually exclusive. Using the addition theorem we have the net probability that the individual will die in the interval $(x_i, x_{i+1})$

$$ q_i, \delta = \int_{x_i}^{x_{i+1}} [\mu(t) - \mu(t;\delta)] dt. $$

Since (2.9) implies that the ratio

$$ \frac{\mu(t) - \mu(t;\delta)}{\mu(t)} = \frac{q_i - 0_i \delta}{q_i} $$

is independent of time $t$, we write

$$ \mu(t) - \mu(t;\delta) = \frac{\mu(t) - \mu(t;\delta)}{\mu(t)} \mu(t) $$

$$ = \frac{q_i - 0_i \delta}{q_i} \mu(t) $$

and

$$ \mu(t) - \mu(t;\delta) = \frac{q_i - 0_i \delta}{q_i} \mu(t). $$

(2.24)

(2.25)
Substituting (2.24) and (2.25) in (2.23) gives

\[ q_{i,\delta} = \int_{x_i}^{x_{i+1}} e^{\int_{x_i}^{x_{i+1}} \mu(t) \, dt} \left( \frac{q_i - q_{i,\delta}}{q_i} \right) \mu(t) \, dt, \quad (2.26) \]

and integrating the right-hand side of (2.26) yields the relationship

\[ q_{i,\delta} = 1 - p_i \left( \frac{q_i - q_{i,\delta}}{q_i} \right). \quad (2.27) \]

Formula (2.27) also can be simplified using Newton's binomial expression as was formula (2.17). Again taking the first four terms of the series, we have

\[ p_i \left( \frac{q_i - q_{i,\delta}}{q_i} \right) = (1-q_i) \left( q_i - q_{i,\delta} \right)/q_i \]

\[ = 1 - (q_i - q_{i,\delta}) - \frac{1}{2} (q_i - q_{i,\delta})^2 \delta \gamma_i + \frac{1}{6} (q_i - q_{i,\delta})^3 \delta \gamma_i \delta (q_i + Q_{i,\delta}). \quad (2.28) \]

Substituting (2.28) in (2.27) and simplifying the resulting expression gives the desired formula

\[ q_{i,\delta} = (q_i - q_{i,\delta}) \left[ 1 + \frac{1}{2} \delta \gamma_i + \frac{1}{6} (q_i + Q_{i,\delta}) \right]. \quad (2.29) \]

Because of the absence of competing risks, the net probability is always greater than the corresponding crude probability, or

\[ q_{i,\delta} > Q_{i,\delta}. \quad (2.30a) \]

Further, if two risks \( R_\delta \) and \( R_\varepsilon \) are such that

\[ Q_{i,\delta} > Q_{i,\varepsilon}, \]

then

\[ q_{i,\delta} > q_{i,\varepsilon} \quad \text{and} \quad q_{i,\delta} < q_{i,\varepsilon}. \quad (2.30b) \]

Verification of (2.30a) and (2.30b) is left to the reader.
2.2. Relation between crude and partial crude probabilities.

Suppose now that risk \( R_1 \) is eliminated from the population. In the presence of all other risks, let \( Q_{i\delta \cdot 1} \) be the partial crude probability that an individual alive at time \( x_i \) will die in the interval \( (x_i, x_{i+1}) \) from cause \( R_\delta \) for \( \delta = 2, \ldots, r \). We shall express \( Q_{i\delta \cdot 1} \) in terms of the probabilities \( p_i \) and \( q_i \) and of the crude probabilities \( Q_{i1} \) and \( Q_{i\delta} \). Using the multiplication and addition theorems as in (2.22) we have

\[
Q_{i\delta \cdot 1} = \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^{t} \left[ \mu(t) - \mu(t;1) \right] dt} \mu(t;\delta) dt .
\]

(2.31)

To simplify (2.31), we note from (2.9) that the ratio \( \frac{\mu(t;\delta)}{[\mu(t) - \mu(t;1)]} \) is equal to \( \frac{Q_{i\delta}}{q_i - Q_{i1}} \) and is independent of time \( t \). The partial crude probability may then be rewritten as

\[
Q_{i\delta \cdot 1} = \frac{\mu(t;\delta)}{\mu(t) - \mu(t;1)} \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^{t} \left[ \mu(t) - \mu(t;1) \right] dt} \mu(t;\delta) dt
\]

\[
= \frac{Q_{i\delta}}{q_i - Q_{i1}} \left[ 1 - e^{-\int_{x_i}^{x_{i+1}} \left[ \mu(t) - \mu(t;1) \right] dt} \right]
\]

(2.32)

Substituting (2.29) for \( \delta = 1 \) in (2.32) gives the final formula:

\[
Q_{i\delta \cdot 1} = Q_{i\delta} \left[ 1 + \frac{1}{2} Q_{i1} + \frac{1}{6} Q_{i1} \left( q_i + Q_{i1} \right) \right], \quad \delta = 2, \ldots, r.
\]

(2.33)
The sum of $Q_{16 \cdot 1}$ for $\delta = 2, \ldots, r$ is equal to the net probability of death when risk $R_1$ is eliminated from the population. That is,

$$\sum_{\delta = 2}^{r} Q_{16 \cdot 1} = \sum_{\delta = 2}^{r} Q_{16}[1 + \frac{1}{2} Q_{11} + \frac{1}{6} Q_{11}(q_{1}+Q_{11})] = (q_{1}+Q_{11})(1 + \frac{1}{2} Q_{11} + \frac{1}{6} Q_{11}(q_{1}+Q_{11}))$$

$$= q_{1} .$$

as might have been anticipated.

Formula (2.33) can be easily generalized to other cases where more than one risk is eliminated. If risks $R_1$ and $R_2$ are eliminated, the partial crude probability that an individual alive at time $x_i$ will die from cause $R_5$ in the interval $(x_i, x_{i+1})$ is:

$$Q_{16 \cdot 12} = Q_{16}[1 + \frac{1}{2} (Q_{11} + Q_{12}) + \frac{1}{6} (Q_{11} + Q_{12})(q_{1}+Q_{11}+Q_{12})] .$$

In the discussion of these three types of probability, both $q_{1}$ and $p_{1}$ are assumed to be greater than zero but less than unity. If $q_{1}$ were zero ($p_{1}=1$), then $Q_{16}$ would also be zero for $\delta = 1, \ldots, r$. Then the ratios $Q_{16}/Q_{4}$, $Q_{16}/(q_{1}+Q_{11})$, and $(q_{1}+Q_{11})/q_{1}$ and formulas (2.17), (2.26), (2.33) and (2.35) would become meaningless. In other words, if an individual were certain to survive an interval, it would be meaningless to speak of his chance of dying from a specific risk. On the other hand, if $p_{1}$ were zero ($q_{1}=1$), formula (2.4) shows that the integral

$$\int_{x_{i}}^{x_{i+1}} \mu(t) \, dt$$

would approach infinity; this fortunately is extremely unrealistic.
3. **Competing Risks with Interaction**

The theory of competing risks presented in Section 2 was based on the independence assumption in (2.2). Under this assumption, the risks act independently of one another and the presence or elimination of one risk has no effect on the intensity functions (forces of mortality) of other risks. The validity of the assumption depends upon the diseases under consideration. One can certainly visualize a situation where the independence assumption does not hold. The presence of tuberculosis ($R_1$), for example, may affect the chance of dying from pneumonia, $R_2$. Once tuberculosis is eliminated as a risk of death, how does one evaluate the probability of dying from pneumonia? The problem can be resolved by creating another risk, $R_{12}$, the interaction between tuberculosis and pneumonia, with the intensity function $\nu(t;1,2)$. When tuberculosis is eliminated as a risk of death, the interaction vanishes also. The purpose of this section is to study the theory of competing risks with the consideration of interactions.

For any two risks $R_0$ and $R_c$, we denote by $R_{0c}$ their interaction and by $\nu(t;\delta,\epsilon)$ the corresponding intensity function, with

$$\nu(t;\delta,\epsilon) \geq 0 \quad (3.2)$$

When two risks have positive interaction, the intensity function $\nu(t;\delta,\epsilon)$ is positive and has the following probabilistic meaning:

$$\nu(t;\delta,\epsilon)\Delta + o(\Delta) = \Pr\{\text{an individual alive at time } t \text{ will die in interval } (t,t+\Delta) \text{ from } R_{0\epsilon}\} \quad (3.3)$$

If two risks have no interaction, $\nu(t;\delta,\epsilon) = 0$. It is conceivable that, for two particular risks, presence of one decreases the probability of dying from the other, so that $\nu(t;\delta,\epsilon) < 0$. In such cases proper interpretation is the following
\[ J(t; \delta) + J(t; \varepsilon) + J(t; \delta, \varepsilon) = \Pr( \text{an individual alive at } t \text{ will die in interval } (t, t+\Delta) \text{ from either } \delta \text{ or } \varepsilon ) \]

For convenience of our discussion, we assume that all \( J(t; \delta, \varepsilon) \) are non-negative.

Under the present framework, the intensity functions satisfy the relation

\[
\sum_{\delta=1}^{r} \mu(t; \delta) + \sum_{\delta=1}^{r-1} \sum_{\varepsilon=\delta+1}^{r} \mu(t; \delta, \varepsilon) = \mu(t). \tag{3.5}
\]

We shall assume, as in Section 2, that the intensity functions \( \mu(t; \delta) \) satisfy condition (2.3) and that the ratio

\[
\frac{\mu(t; \delta, \varepsilon)}{\mu(t)} = c_{i \delta \varepsilon}, \text{ for } x_i \leq t < x_{i+1}, \tag{3.6}
\]

is independent of time \( t \), but is a function of the interval \((x_i, x_{i+1})\) and risks \( R_0 \) and \( R_\varepsilon \). In this case the formulas for the probability \( p_i(q_i) \), the crude probability \( q_{i0} \), and the net probability \( q_{i\delta} \) all remain the same as those in Section 2. Namely,

\[
p_i = e^{-\int_{x_i}^{x_{i+1}} \mu(t) dt}, \quad q_i = 1 - p_i \tag{2.4}
\]

\[
q_{i0} = \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^{x'} \mu(t) dt} \mu(t; \delta) dt \tag{2.6}
\]

and

\[
q_{i\delta} = \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^{x'} \mu(t; \delta) dt} \mu(t; \delta) dt \tag{2.15}
\]

so that the relation,
and

\[ q_{i0} = 1 - p_{i}^{Q_{i0}/q_{i}} \]  \hspace{1cm} (2.17)

and

\[ q_{i0} = Q_{i0}[1 + \frac{1}{2}(q_{i} - Q_{i0}) + \frac{1}{6}(q_{i} - Q_{i0})(2q_{i} - 2Q_{i0})] \]  \hspace{1cm} (2.21)

also holds. Corresponding to the interaction \( R_{i0} \), there is the crude probability of dying from \( R_{i0} \) in the presence of all risks,

\[ Q_{i0} = \int_{x_{i}}^{x_{i+1}} e^{-\int_{x_{i}}^{t} \mu(t) dt} \mu(t, \delta, \epsilon) dt . \]  \hspace{1cm} (3.7)

In view of the proportionality assumption in (3.6), we may rewrite (3.7) as follows

\[ Q_{i0} = \frac{\mu(t; \delta, \epsilon)}{\mu(t)} \int_{x_{i}}^{x_{i+1}} e^{-\int_{x_{i}}^{t} \mu(t) dt} \mu(t, \delta, \epsilon) dt \]  \hspace{1cm} (3.7a)

When risk \( R_{i} \) is removed as a cause of death, its interactions with all other causes, \( R_{i2}, \ldots, R_{im} \), will all vanish, and the net probability of dying in \((x_{i}, x_{i+1})\) is given by

\[ q_{i} = 1 - e^{-\int_{x_{i}}^{x_{i+1}} [\mu(t) - \mu(t, 1) - \sum_{\epsilon=2}^{r} \mu(t, 1, \epsilon)] dt} \]  \hspace{1cm} (3.8)

Using the proportionality assumption (2.3) and (3.6), the exponent in (3.8) may be rewritten

\[ \int_{x_{i}}^{x_{i+1}} [\mu(t) - \mu(t, 1) - \sum_{\epsilon=2}^{r} \mu(t, 1, \epsilon)] dt = [\int_{x_{i}}^{x_{i+1}} \mu(t) dt] - \frac{\mu(t) - \mu(t, 1)}{\mu(t)} - \sum_{\epsilon=2}^{r} \mu(t, 1, \epsilon) \]

which, because of (2.9) and (3.7a), becomes
Substituting (3.9) in (3.8) yields a relationship between the net probability $q_{1.1}$ and $q_i$ and the crude probabilities:

$$q_{1.1} = \frac{(q_i - Q_{i1} - \Sigma Q_{i1e})/q_i}{1 - p_i}$$

(3.10)

where the summation $\Sigma Q_{i1e}$ in the exponent is taken over $\epsilon = 2, \ldots, r$.

Applying the binomial expression to the last term in (3.10) and taking the first three terms of the infinite series as in (2.17), we have

$$q_{1.1} = (1 - (q_i - Q_{i1} - \Sigma Q_{i1e})/q_i - 1/2(q_i - Q_{i1} - \Sigma Q_{i1e})(Q_{i1} + \Sigma Q_{i1e}) - 1/6(q_i - Q_{i1} - \Sigma Q_{i1e})(Q_{i1} + \Sigma Q_{i1e})(q_i + Q_{i1} + \Sigma Q_{i1e})$$

(3.11)

and the approximate formula 4/

$$q_{1.1} = (q_i - Q_{i1} - \Sigma Q_{i1e}) \left[ 1 + \frac{1}{2} \left( Q_{i1} + \Sigma Q_{i1e} \right) + \frac{1}{6} \left( Q_{i1} + \Sigma Q_{i1e} \right) \left( q_i + Q_{i1} + \Sigma Q_{i1e} \right) \right]$$

(3.12)

The partial crude probability of dying from $R_0$ when $R_1$ is eliminated can be obtained in a similar manner. Corresponding to formula (2.31) we now have

$$Q_{1\delta,1} = \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^t \left[ \mu(t) - \mu(t;1) - \Sigma \mu(t;1,\epsilon) \right] dt} \mu(t;\delta) dt$$

$$= \frac{\mu(t;\delta)}{\mu(t) - \mu(t;1) - \Sigma \mu(t;1,\epsilon)} \int_{x_i}^{x_{i+1}} e^{-\int_{x_i}^t \left[ \mu(t) - \mu(t;1) - \Sigma \mu(t;1,\epsilon) \right] dt} \left[ \mu(t) - \mu(t;1) - \Sigma \mu(t;1,\epsilon) \right] dt$$

(3.13)
where

\[ \frac{u(t;\epsilon)}{u(t) - u(t;\epsilon) - \sum_{\epsilon} u(t;\epsilon,\epsilon)} = \frac{Q_{i\delta}}{q_i - Q_{i1} - \sum_{\epsilon=2}^\infty Q_{i1\epsilon}} \]  \hspace{1cm} (3.14)

and

\[ x_{i+1} = \int_{x_i}^{x_i+1} \left[ u(t) - u(t;\epsilon) - \sum_{\epsilon} u(t;\epsilon,\epsilon) \right] dt \]

\[ \int_{x_i}^{x_i+1} [u(t) - u(t;\epsilon) - \sum_{\epsilon} u(t;\epsilon,\epsilon)] dt = \frac{(q_i - Q_{i1} - \sum_{\epsilon=2}^\infty Q_{i1\epsilon})}{q_i} = q_{i1} \cdot \] \hspace{1cm} (3.15)

Therefore

\[ Q_{i\delta1} = \frac{Q_{i\delta}}{q_i - Q_{i1} - \sum_{\epsilon=2}^\infty Q_{i1\epsilon}} q_{i1} \cdot \] \hspace{1cm} (3.16)

In the following appendix we shall present the problem of estimation and the multiple decrement tables without considering the interaction. The case where the interactions are present is completely analogous.
1/ When the first three terms of the binomial series are taken, we have
\[ q_{i.\delta} = (q_{i} - Q_{i\delta})[1 + \frac{1}{2} Q_{i\delta}] \quad (2.29a) \]

2/ When formula (2.29a) in footnote 1 is used, we have
\[ Q_{i\delta.1} = Q_{i\delta}[1 + \frac{1}{2} Q_{i1}] \quad (2.33a) \]

3/ Corresponding to formula (2.33a) in footnote 2, we have
\[ Q_{i\delta.12} = Q_{i\delta}[1 + \frac{1}{2}(Q_{i1} + Q_{i2})] \quad (2.35a) \]

4/ Corresponding to formula (2.29a) in footnote 1, we have
\[ q_{i.1} = (q_{i} - Q_{i1} - \frac{\xi}{2} Q_{i1\xi})[1 + \frac{1}{2} (Q_{i1} + \frac{\xi}{2} Q_{i1\xi})]. \quad (3.12a) \]
APPENDIX IV

MULTIPLE DECREMENT TABLES

1. Introduction

In studies of competing risks in a given population, deaths are classified according to cause. The number of deaths from each specific cause is the basic random variable for estimating the corresponding probability and for making inference about the population in question. The statistical theory involves the multinomial distribution, which is described as follows.

Suppose that \( t \) individuals alive at the beginning of an age interval \((x_i, x_{i+1})\) are subject to \( r \) risks of death, \( R_1, \ldots, R_r \), with the corresponding probabilities \( Q_1, \ldots, Q_r \), respectively. Let \( d_i \) be the number of deaths occurring in the interval from \( R_0 \) so that the sum

\[
d_{i1} + \cdots + d_{ir} = d_i
\]  

(1.1)

is the total number of deaths, and the difference

\[
\ell_i - d_i = \ell_{i+1}
\]  

(1.2)

is the number of survivors at the end of the interval. This means that

\[
\ell_i = d_{i1} + \cdots + d_{ir} + \ell_{i+1}.
\]  

(1.3)

The corresponding probabilities have a similar relationship (cf. equation (2.10), Appendix III)

\[
Q_{i1} + \cdots + Q_{ir} = q_i;
\]  

(1.4)

and the difference

\[
1 - q_i = p_i
\]  

(1.5)
is the probability of survival, so that

\[ 1 = Q_{i1} + \cdots + Q_{ir} + p_i. \]  

Equations (1.3) and (1.6) define a multinomial distribution, where the random variables \(d_{i1}, \ldots, d_{ir}\) and \(\ell_{i+1}\) have the joint probability:

\[ \frac{\ell_i!}{d_{i1}! \cdots d_{ir}! \ell_{i+1}!} Q_{i1} d_{i1} \cdots Q_{ir} d_{ir} p_i^{\ell_{i+1}} \]  

(1.7)

where \(d_{i1} + \cdots + d_{ir} + \ell_{i+1} = \ell_i\).

In formula (1.7) the quantity \(Q_{i1} d_{i1} \cdots Q_{ir} d_{ir} p_i^{\ell_{i+1}}\) is the probability that the specified individual will die from \(R_\delta\), for \(\delta = 1, \ldots, r\), and the remaining \(\ell_{i+1}\) individuals will survive the interval \((x_i, x_{i+1})\). The combinatorial factor

\[ \frac{\ell_i!}{d_{i1}! \cdots d_{ir}! \ell_{i+1}!} \]

is the number of possibilities that \(d_{i\delta}\) people among \(\ell_i\) will die from \(R_\delta\) and \(\ell_{i+1}\) will survive. The expected values, the variances and the covariances of the distribution can be obtained from (1.7) directly. However, the following approach is somewhat simpler.

Each \(d_{i\delta}\) is in effect a binomial random variable in \(\ell_i\) "trials" with the binomial probability \(Q_{i\delta}\). Therefore, the expected value and the variance of \(d_{i\delta}\) are given by

\[ E(d_{i\delta} | \ell_i) = \ell_i Q_{i\delta} \]  

(1.8)

and

\[ \text{Var}(d_{i\delta} | \ell_i) = \ell_i Q_{i\delta} (1 - Q_{i\delta}), \delta = 1, \ldots, r. \]  

(1.9)

---

(1) For simplicity of formulas, no symbols for the values that the random variables \(\ell_i\) and \(d_{i\delta}\) assume are introduced in this Appendix.
The covariance between any two random variables \( d_{i\delta} \) and \( d_{i\varepsilon} \) is

\[
\text{Cov}(d_{i\delta}, d_{i\varepsilon}) = -\frac{\lambda_1 Q_{i\delta} Q_{i\varepsilon}}{\sqrt{\lambda_1 Q_{i\delta} (1-Q_{i\delta}) \lambda_1 Q_{i\varepsilon} (1-Q_{i\varepsilon})}}.
\]

Therefore, the correlation coefficient between \( d_{i\delta} \) and \( d_{i\varepsilon} \) is negative.

\[
\rho_{d_{i\delta}, d_{i\varepsilon}} = -\frac{\lambda_1 Q_{i\delta} Q_{i\varepsilon}}{\sqrt{(1-Q_{i\delta})(1-Q_{i\varepsilon})}}.
\]

Formula (1.11) shows that the larger the probabilities \( Q_{i\delta} \) and \( Q_{i\varepsilon} \), the greater will be the correlation coefficient in absolute value. Thus the greater the number of deaths from one cause, the smaller will be the number of deaths from another cause; and the two risks, \( R_\delta \) and \( R_\varepsilon \), are indeed competing risks.

The covariance between the number of deaths from a specific cause \( d_{i\delta} \) and the number of survivors \( \lambda_{i+1} \) can be obtained in a similar manner. The formula is

\[
\text{Cov}(d_{i\delta}, \lambda_{i+1}) = -\frac{\lambda_1 Q_{i\delta} \lambda_1}{\sqrt{\lambda_1 Q_{i\delta} (1-Q_{i\delta}) \lambda_1 (1-Q_{i\delta})}}.
\]

and hence the correlation coefficient

\[
\rho_{d_{i\delta}, \lambda_{i+1}} = -\sqrt{\frac{Q_{i\delta} \lambda_1}{(1-Q_{i\delta})(1-\lambda_1)}}.
\]

which also increases in absolute value with \( Q_{i\delta} \) and \( \lambda_1 \). This means that the greater the number of deaths from one cause, the fewer will be the survivors.

When \( \lambda_1 \) is a given fixed number, the variance of \( \lambda_1 \) is zero. Applying the formula of the variance of a sum of random variables to formula (1.3) we have
Substituting formulas (1.9), (1.10), (1.12) and the conditional variance of $\ell_{i+1}$ given $\ell_i$,

$$\text{Var}(\ell_{i+1} | \ell_i) = \sigma_i^2 p_i q_i,$$  \hspace{1cm} (1.15)

in (1.14) yields a relationship

$$\sum_{\delta=1}^{r} \ell_i Q_{i\delta} (1 - Q_{i\delta}) + \ell_i p_i q_i - 2 \sum_{\delta=1}^{r-1} \sum_{\epsilon=\delta+1}^{r} \ell_i Q_{i\delta} Q_{i\epsilon}$$

$$- 2 \sum_{\delta=1}^{r} \ell_i Q_{i\delta} p_i = 0 \hspace{1cm} (1.16)$$

Verification of (1.16) is left to the reader.
2. The Chain Multinomial Distributions

In the preceding section we were concerned with the probability distribution of the number of deaths occurring in interval \((x_i, x_{i+1})\) based on \(z_i\) people alive at age \(x_i\). When we start at age \(x_0\) with \(z_0\) individuals, the number of survivors \(l_1, l_2, \ldots\) at ages \(x_1, x_2\) are themselves random variables. The probability distribution of each \(l_i\) is dependent upon the number of survivors of the preceding interval, for \(i=1,2\ldots\). As a result, we have a chain of multinomial distributions. In other words, for any positive integer, \(u\), the joint probability distribution of all the random variables \(d_{i1}, \ldots, d_{ir}, l_{i+1}\), for \(i=0,1,\ldots,u\) is

\[
\frac{\prod_{i=0}^{u} l_i!}{d_{i1}! \cdots d_{ir}! l_{i+1}!} \cdot q_{i1} \cdots q_{ir} p_{i}^{l_{i+1}} \quad (2.1)
\]

with \(d_{i1}, \ldots, d_{ir}, l_{i+1}\) being non-negative integers and satisfying the restriction

\[d_{i1} + \ldots + d_{ir} + l_{i+1} = l_i.
\]

The expected values and variances of the random variables \(d_{i1}\) may be derived from those obtained in Section 1 by using the rule on conditional expectation and conditional variance. The expectation of \(d_{i1}\) is the expectation of the condition expectation of \(d_{i1}\) given \(l_i\)

\[
E(d_{i1}) = E[E(d_{i1} | l_i)] = E[l_i q_{i1}]
\]

\[= E(l_i) q_{i1} = l_i p_{01} q_{i1}, \quad (2.2)
\]

where

\[p_{01} = \exp\left(-\int_0^x \mu(t) dt\right)\]
is the probability of surviving from $x_0$ to $x_i$.

The rule on the variances is a little more complex. When $\lambda_1$ is a random variable, the conditional variance of $d_{i\delta}$ given $\lambda_1$ is also a random variable and has an expectation

$$E[\text{Var}(d_{i\delta}|\lambda_1)] = E[\lambda_0^2 \delta_1 Q_1 (1-Q_1)]$$

$$= E(\lambda_1) Q_1 (1-Q_1) = \lambda_0 p_{01} Q_1 (1-Q_1) \quad (2.3)$$

On the other hand, the conditional expectation $E(d_{i\delta}|\lambda_1)$, being a random variable, has the variance

$$\text{Var}[E(d_{i\delta}|\lambda_1)] = \text{Var}[\lambda_0^2 (1-p_{01})]$$

$$= \lambda_0^2 Q_1 (1-p_{01}) \quad (2.4)$$

According to the rule, the variance of $d_{i\delta}$ is given by

$$\text{Var}(d_{i\delta}) = E[\text{Var}(d_{i\delta}|\lambda_1)] + \text{Var}[E(d_{i\delta}|\lambda_1)] \quad (2.5)$$

Substituting (2.3) and (2.4) in (2.5) and simplifying the resulting expression yield the formula

$$\text{Var}(d_{i\delta}) = \lambda_0 p_{01} Q_1 (1-P_{01} Q_1), \quad i=0,\ldots,u \quad (2.6)$$

Regarding the covariance of $d_{i\delta}$ and $d_{1\epsilon}$, the rule is

$$\text{Cov}(d_{i\delta},d_{1\epsilon}) = E[\text{Cov}(d_{i\delta},d_{1\epsilon}|\lambda_1)] + \text{Cov}[E(d_{i\delta}|\lambda_1), E(d_{1\epsilon}|\lambda_1)]$$

and hence the formula for the covariance is

$$\text{Cov}(d_{i\delta},d_{1\epsilon}) = -\lambda_0 p_{01} Q_1 (1-P_{01} Q_1), \quad \delta \neq \epsilon; \quad \delta, \epsilon = 1,\ldots,r; \quad i=0,\ldots,u \quad (2.7)$$

Formulas (2.6) and (2.7) can be justified intuitively. An individual alive at $x_0$ has a probability $P_{01} Q_1$ of dying from $R_{01}$ in interval $(x_i, x_{i+1})$. The
number of individuals dying from R₀, d₁₀, has a binomial distribution in ₀ "trials" with the probability p₀₁Q₁₀. It follows from the binomial theory that the variance of d₁₀ is given by (2.6) [Cf. formula (1.9)]. Similarly, the random variables d₁₀ and d₁ₑ have a joint multinomial distribution with the corresponding probabilities p₀₁Q₁₀ and p₀₁Q₁ₑ, respectively. Therefore, their covariance is given by

\[ \text{Cov}(d₁₀, d₁ₑ) = -2₀₀₁Q₁₀Q₁ₑ \]  

and their correlation coefficient is

\[ \rho_{d₁₀, d₁ₑ} = -\frac{p₀₁Q₁₀Q₁ₑ}{\sqrt{1 - p₀₁Q₁₀} \sqrt{1 - p₀₁Q₁ₑ}} \]  

(2.9)

The negative correlation coefficient again indicates the competition between two risks, and the negative correlation is more pronounced when the corresponding probabilities of death, Q₁₀ and Q₁ₑ, are large. Also the correlation coefficient increases in absolute value with p₀ᵢ, the probability of surviving the interval (x₀, x₁). Since p₀ᵢ decreases with x₁, the competition between two risks is more acute at young ages or when the probability of dying Q₁₀ (Q₁ₑ) is large.

For the numbers of deaths occurring in two different age intervals (xᵢ, xᵢ₊₁) and (xᵣ, xᵣ₊₁), the corresponding covariance is obtained by using once again the fact that the random variables d₁₀ and dᵣₑ have a joint multinomial distribution in ₀ "trials" with the corresponding probabilities p₀ᵢQᵢ and p₀ᵣQᵣₑ so that

\[ \text{Cov}(d₁₀, dᵣₑ) = -2₀₀ᵢQ₁₀Qᵣₑ \]  

(2.10)

and the correlation coefficient

\[ \rho_{d₁₀, dᵣₑ} = -\frac{p₀ᵢQ₁₀Qᵣₑ}{\sqrt{1 - p₀ᵢQ₁₀} \sqrt{1 - p₀ᵣQᵣₑ}} \]  

(2.11)

is again negative. Without loss of generality, we may assume that i < r and
use the relationship \( p_{0j} = p_{0i} p_{ij} \) and write

\[
\rho_{d_{i \delta}, d_{j \varepsilon}} = - p_{0i} \sqrt{\frac{p_{ij}}{1 - p_{0i}}} \sqrt{\frac{Q_{i \delta}}{1 - p_{0j} Q_{j \varepsilon}}} \]  

(2.12)

which differs from the correlation coefficient in (2.9) by the factor \( \sqrt{p_{ij}} \), and equals (2.9) when \( j = i \). Thus the correlation coefficient between \( d_{i \delta} \) and \( d_{j \varepsilon} \) is generally smaller in absolute value than the correlation coefficient between \( d_{i \delta} \) and \( d_{i \varepsilon} \). For a fixed \( x_i \), the probability \( p_{ij} \) of surviving from \( x_i \) to \( x_j \) decreases as \( x_j \) increases. This means that the competition between risks at two different ages diminishes as the two ages become more distant.

The covariance between the number dying and the number surviving can be derived in a similar way:

\[
\text{Cov}(d_{i \delta}, \ell_{j}) = - \ell_0 p_{0i} Q_{i \delta} p_{0j} \]  

(2.13)

and

\[
\text{Cov} (\ell_i, d_{j \delta}) = \ell_0 (1-p_{0i}) p_{0j} Q_{j \delta}, \quad \delta = 1, \ldots, r; \quad i < j; \quad i, j = 0, 1, \ldots. \]  

(2.14)

It is interesting to note that the covariance between \( \ell_i \) and \( d_{j \delta} \) in (2.14) is the only one carrying a positive sign. The positive covariance in (2.14) indicates that the larger the number of survivors at age \( x_i \), the greater the probability that a larger number of deaths from \( R_\delta \) will occur in a subsequent interval \((x_j, x_{j+1})\). The covariance between \( \ell_i \) and \( \ell_j \)

\[
\text{Cov} (\ell_i, \ell_j) = \ell_0 (1-p_{0i}) p_{0j}, \quad i < j. \]  

(2.15)

has been given in (3.7) in Appendix II. These results show that, for each \( u \), the random variables \( d_{i \delta} \) and \( \ell_{i+1} \), for \( i = 0, 1, \ldots, u; \ \delta = 1, \ldots, r \), have a chain of multinomial distributions with the probability distribution given in (2.1) and the expectations and covariances given in (2.2) through (2.15).
3. Estimation of the Crude Probabilities

The estimators of the crude probabilities $Q_{ik}$ and $p_i$ can be derived directly from the joint probability function

$$
L = \prod_{i=0}^{u} \frac{\ell_i!}{d_{i1}! \cdots d_{ir}! q_{i+1}!} Q_{i1}^{d_{i1}} \cdots Q_{ir}^{d_{ir}} p_i^{\ell_{i+1}} \tag{3.1}
$$

by using the maximum likelihood principle.

The logarithm of the likelihood functions is

$$
\log L = k + \sum_{i=0}^{u} d_{i1} \log Q_{i1} + \cdots + d_{ir} \log Q_{ir} + \ell_{i+1} \log p_i \tag{3.2}
$$

where $k$ is constant and the probabilities are not all independent but satisfy the relationship for each $i$

$$
Q_{i1} + \cdots + Q_{ir} + p_i = 1 \tag{3.3}
$$

The estimators $\hat{Q}_{i1}, \ldots, \hat{Q}_{ir}$, and $\hat{p}_i$ are the maximizing values of $\log L$ subject to condition (3.3). Using the Lagrange method we maximize

$$
\phi = k + \sum_{i=0}^{u} \sum_{\delta=1}^{r} d_{i\delta} \log Q_{i\delta} + \ell_{i+1} \log p_i - \lambda_i \left( \sum_{\delta=1}^{r} Q_{i\delta} + p_i - 1 \right) \tag{3.3}
$$

Differentiating $\phi$ with respect to $Q_{i1}, \ldots, Q_{ir}, p_i$ and setting the derivatives equal to zero, we have the following simultaneous equations

$$
\frac{\partial \phi}{\partial Q_{i\delta}} = \frac{d_{i\delta}}{\lambda_i} - \lambda_i = 0 \quad \text{or} \quad \hat{Q}_{i\delta} = \frac{d_{i\delta}}{\lambda_i}, \quad \delta=1, \ldots, r \tag{3.4}
$$

$$
\frac{\partial \phi}{\partial p_i} = \frac{\ell_{i+1}}{p_i} - \lambda_i = 0 \quad \text{or} \quad \hat{p}_i = \frac{\ell_{i+1}}{\lambda_i} \tag{3.5}
$$

For each $i$, there are $r+2$ equations in (3.3), (3.4), and (3.5) with $r+2$ unknowns: $\hat{Q}_{i1}, \ldots, \hat{Q}_{ir}, \hat{p}_i$ and $\lambda_i$, where $\lambda_i$ is known as the Lagrange coefficient. To solve these equations simultaneously we substitute (3.4) and (3.5) in (3.3),
and use (3.6) in (3.4) and (3.5) to obtain the maximum likelihood estimators

\[ \hat{Q}_{i\delta} = \frac{d_{i\delta}}{\lambda_i}, \quad \delta = 1, \ldots, r \quad i = 0, \ldots, u \] (3.7)

and

\[ \hat{\rho}_i = \frac{\lambda_{i+1}}{\lambda_i}, \quad i = 0, \ldots, u \] (3.8)

On the right-hand side of (3.7) and (3.8) are the proportions dying in the interval \((x_i, x_{i+1})\) from \(R_\delta\) and the proportion surviving the interval, respectively. Thus the maximum likelihood estimators derived (3.7) and (3.8) are consistent with our intuition. Further, they are unbiased estimators, since their expected values are equal to the corresponding probabilities. This is demonstrated below.

\[ E[\hat{Q}_{i\delta}] = E[\frac{d_{i\delta}}{\lambda_i}] = E[E(d_{i\delta} | \lambda_i) \frac{1}{\lambda_i}] \] (3.9)

where the conditional expectation is given in Section 1,

\[ E(d_{i\delta} | \lambda_i) = \lambda_i Q_{i\delta} \] (1.8)

so that

\[ E[\hat{Q}_{i\delta}] = Q_{i\delta} \] (3.10)
as required to be shown.

The variances and covariances of the estimators can be computed directly.

For the estimator \( \hat{Q}_{1\delta} \),

\[
\text{Var}(\hat{Q}_{1\delta}) = E[\hat{Q}_{1\delta}^2] - Q_{1\delta}^2,
\]

where

\[
E[\hat{Q}_{1\delta}^2] = E\left[\frac{d_{i\delta}^2}{\varphi_{i\delta}^2}\right] = E\left[E\left(d_{i\delta}^2 \mid \varphi_{i\delta}\right) \frac{1}{\varphi_{i\delta}^2}\right].
\]

We recall from Section 1 that, given \( \varphi_{i\delta} \), the number of deaths \( d_{i\delta} \) is a binomial random variable having the variance

\[
\text{Var}(d_{i\delta} \mid \varphi_{i\delta}) = \varphi_{i\delta} Q_{i\delta}(1-Q_{i\delta}),
\]

and the expectation of the square

\[
E(d_{i\delta}^2 \mid \varphi_{i\delta}) = \varphi_{i\delta} Q_{i\delta}(1-Q_{i\delta}) + \varphi_{i\delta}^2 Q_{i\delta}^2.
\]

Consequently, the expectation in (3.12) may be rewritten

\[
E[\hat{Q}_{1\delta}^2] = E\left(\frac{1}{\varphi_{i\delta}}\right) Q_{i\delta}(1-Q_{i\delta}) + Q_{1\delta}^2.
\]

Substituting (3.14) in (3.11) gives the formula

\[
\text{Var}(\hat{Q}_{1\delta}) = E\left(\frac{1}{\varphi_{i\delta}}\right) Q_{i\delta}(1-Q_{i\delta}), \quad \delta=1,\ldots,r
\]

Using a similar approach, we obtain

\[
\text{Var}(\hat{p}_i) = \text{Var}(\hat{q}_i) = E\left(\frac{1}{\varphi_i}\right) p_i q_i.
\]

\[
\text{Cov}(\hat{Q}_{1\delta}, \hat{Q}_{1\epsilon}) = -E\left(\frac{1}{\varphi_{i\delta}}\right) Q_{i\delta} Q_{i\epsilon},
\]

and

\[
\text{Cov}(\hat{Q}_{1\delta}, \hat{p}_i) = -E\left(\frac{1}{\varphi_{i\delta}}\right) p_i Q_{i\delta}.
\]
When the original cohort \( l_0 \) is large, the expectation of the reciprocal of 
\( l_1 \) may be approximated by the reciprocal of the expectation. That is

\[
E\left( \frac{1}{l_1} \right) \approx \frac{1}{E(l_1)} = \frac{1}{\xi_0 p_0} . \tag{3.19}
\]

In order to make inferences concerning the crude probabilities, it is necessary to find the sample estimates of the standard errors of \( \hat{q}_{i0} \). This may be done by substituting \( \hat{q}_{i0} \), \( \hat{p}_1 \), and \( \hat{z}_1 \) for the corresponding unknown parameters in (3.15) and (3.16) and taking the square root of the resulting formulas. Thus,

\[
S_{\hat{q}_{i0}} = \sqrt{\frac{1}{\hat{q}_1} \hat{q}_{i0} (1 - \hat{q}_{i0})} , \quad \delta = 1, \ldots, r \tag{3.20}
\]

and

\[
S_{\hat{q}_i} = \sqrt{\frac{1}{\hat{q}_i} \hat{q}_i (1 - \hat{q}_i)} \quad i = 1, \ldots, u . \tag{3.21}
\]

The main results obtained in this Appendix may be summarized in the following table.

### Table 1. Multiple Decremental Table

<table>
<thead>
<tr>
<th>Age Interval (years)</th>
<th>Number Living at Age ( x_1 )</th>
<th>Proportion Dying in Interval ( (x_i, x_{i+1}) )</th>
<th>Proportions Dying in ( (x_i, x_{i+1}) ) by Causes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \hat{q}<em>i ) ( S</em>{\hat{q}_i} )</td>
<td>( \hat{q}<em>{i1} ) ( S</em>{\hat{q}<em>{i1}} ) ( \ldots ) ( \hat{q}</em>{ir} ) ( S_{\hat{q}_{ir}} )</td>
</tr>
<tr>
<td>( x_i - x_{i+1} )</td>
<td>( l_i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_0 - x_1 )</td>
<td>( l_0 )</td>
<td>( \hat{q}<em>0 ) ( S</em>{\hat{q}_0} )</td>
<td>( \hat{q}<em>{o1} ) ( S</em>{\hat{q}<em>{o1}} ) ( \ldots ) ( \hat{q}</em>{or} ) ( S_{\hat{q}_{or}} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Fraction of Last Age Interval of Life $a_i$

**Table 1**

**Austria, 1969**

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.12</td>
<td>.12</td>
<td>.12</td>
</tr>
<tr>
<td>1-5</td>
<td>.37</td>
<td>.37</td>
<td>.37</td>
</tr>
<tr>
<td>5-10</td>
<td>.47</td>
<td>.47</td>
<td>.47</td>
</tr>
<tr>
<td>10-15</td>
<td>.51</td>
<td>.51</td>
<td>.49</td>
</tr>
<tr>
<td>15-20</td>
<td>.58</td>
<td>.58</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.48</td>
<td>.49</td>
<td>.48</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.50</td>
<td>.54</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.52</td>
<td>.51</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.50</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.48</td>
<td>.47</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.45</td>
<td>.44</td>
<td>.45</td>
</tr>
<tr>
<td>90-95</td>
<td>.40</td>
<td>.40</td>
<td>.40</td>
</tr>
</tbody>
</table>
Table 2

Fraction of Last Age Interval of Life, $a_i$
California, 1960

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Fraction of Last Age Interval of Life $a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.10</td>
</tr>
<tr>
<td>1-5</td>
<td>.39</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.57</td>
</tr>
<tr>
<td>15-20</td>
<td>.57</td>
</tr>
<tr>
<td>20-25</td>
<td>.49</td>
</tr>
<tr>
<td>25-30</td>
<td>.50</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.51</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.46</td>
</tr>
<tr>
<td>90-95</td>
<td>.40</td>
</tr>
</tbody>
</table>
Table 3
Fraction of Last Age Interval of Life, $a_i$
Canada, 1968

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.11</td>
<td>.11</td>
<td>.12</td>
</tr>
<tr>
<td>1-5</td>
<td>.41</td>
<td>.42</td>
<td>.40</td>
</tr>
<tr>
<td>5-10</td>
<td>.45</td>
<td>.45</td>
<td>.44</td>
</tr>
<tr>
<td>10-15</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>15-20</td>
<td>.57</td>
<td>.59</td>
<td>.53</td>
</tr>
<tr>
<td>20-25</td>
<td>.48</td>
<td>.47</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.50</td>
<td>.49</td>
<td>.53</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>80-85</td>
<td>.50</td>
<td>.49</td>
<td>.51</td>
</tr>
<tr>
<td>85-90</td>
<td>.47</td>
<td>.46</td>
<td>.48</td>
</tr>
</tbody>
</table>
Table 4
Fraction of Last Age Interval of Life, $a_1$
Costa Rica, 1963

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i - x_{i+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>.28</td>
<td>.27</td>
<td>.28</td>
</tr>
<tr>
<td>1-5</td>
<td>.29</td>
<td>.29</td>
<td>.28</td>
</tr>
<tr>
<td>5-10</td>
<td>.40</td>
<td>.42</td>
<td>.38</td>
</tr>
<tr>
<td>10-15</td>
<td>.49</td>
<td>.50</td>
<td>.50</td>
</tr>
<tr>
<td>15-20</td>
<td>.55</td>
<td>.55</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>25-30</td>
<td>.53</td>
<td>.51</td>
<td>.55</td>
</tr>
<tr>
<td>30-35</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>35-40</td>
<td>.49</td>
<td>.51</td>
<td>.48</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.54</td>
<td>.52</td>
</tr>
<tr>
<td>45-50</td>
<td>.53</td>
<td>.51</td>
<td>.55</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.55</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.55</td>
<td>.51</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.56</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.52</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.50</td>
<td>.50</td>
<td>.50</td>
</tr>
</tbody>
</table>
Table 5
Fraction of Last Age Interval of Life, $a_i$
Finland, 1968

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.09</td>
<td>.08</td>
<td>.09</td>
</tr>
<tr>
<td>1-5</td>
<td>.38</td>
<td>.41</td>
<td>.34</td>
</tr>
<tr>
<td>5-10</td>
<td>.49</td>
<td>.48</td>
<td>.49</td>
</tr>
<tr>
<td>10-15</td>
<td>.52</td>
<td>.53</td>
<td>.50</td>
</tr>
<tr>
<td>15-20</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.52</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.52</td>
<td>.48</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.55</td>
<td>.54</td>
<td>.55</td>
</tr>
<tr>
<td>45-50</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.49</td>
<td>.52</td>
</tr>
<tr>
<td>80-85</td>
<td>.47</td>
<td>.47</td>
<td>.48</td>
</tr>
<tr>
<td>Age Interval</td>
<td>Fraction of Last Age Interval of Life, $a_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------------</td>
<td>------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Both Sexes</td>
<td>Male</td>
<td>Female</td>
</tr>
<tr>
<td>0-1</td>
<td>.16</td>
<td>.15</td>
<td>.17</td>
</tr>
<tr>
<td>1-5</td>
<td>.38</td>
<td>.39</td>
<td>.36</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
<td>.47</td>
<td>.45</td>
</tr>
<tr>
<td>10-15</td>
<td>.54</td>
<td>.55</td>
<td>.52</td>
</tr>
<tr>
<td>15-20</td>
<td>.56</td>
<td>.56</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.50</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
<td>.48</td>
<td>.50</td>
</tr>
<tr>
<td>85-90</td>
<td>.46</td>
<td>.45</td>
<td>.47</td>
</tr>
<tr>
<td>90-95</td>
<td>.41</td>
<td>.39</td>
<td>.42</td>
</tr>
</tbody>
</table>
Table 7
Fraction of Last Age Interval of Life, $a_i$
East Germany, 1967

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Fraction of Last Age Interval of Life $a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i - x_{i+1}$</td>
<td>Both Sexes</td>
</tr>
<tr>
<td>0-1</td>
<td>.38</td>
</tr>
<tr>
<td>1-5</td>
<td>.46</td>
</tr>
<tr>
<td>5-10</td>
<td>.52</td>
</tr>
<tr>
<td>10-15</td>
<td>.56</td>
</tr>
<tr>
<td>15-20</td>
<td>.50</td>
</tr>
<tr>
<td>20-25</td>
<td>.52</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.48</td>
</tr>
<tr>
<td>85-90</td>
<td>.43</td>
</tr>
<tr>
<td>90-95</td>
<td>.39</td>
</tr>
</tbody>
</table>
Table 8
Fraction of Last Age Interval of Life, $a_i$
West Germany, 1969

<table>
<thead>
<tr>
<th>Age Interval $x_i-x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.10</td>
<td>.10</td>
<td>.11</td>
</tr>
<tr>
<td>1-5</td>
<td>.39</td>
<td>.39</td>
<td>.38</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
<td>.46</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.52</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>15-20</td>
<td>.57</td>
<td>.58</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.54</td>
<td>.55</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>50-55</td>
<td>.58</td>
<td>.58</td>
<td>.57</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.49</td>
<td>.52</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
<td>.47</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.44</td>
<td>.43</td>
<td>.45</td>
</tr>
<tr>
<td>90-95</td>
<td>.39</td>
<td>.38</td>
<td>.40</td>
</tr>
</tbody>
</table>
### Table 9
Fraction of Last Age Interval of Life, $a_i$

**Hungary, 1967**

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both $a_i$</th>
<th>Male $a_i$</th>
<th>Female $a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0.10</td>
<td>0.10</td>
<td>0.11</td>
</tr>
<tr>
<td>1-5</td>
<td>0.35</td>
<td>0.35</td>
<td>0.33</td>
</tr>
<tr>
<td>5-10</td>
<td>0.45</td>
<td>0.47</td>
<td>0.42</td>
</tr>
<tr>
<td>10-15</td>
<td>0.52</td>
<td>0.51</td>
<td>0.54</td>
</tr>
<tr>
<td>15-20</td>
<td>0.55</td>
<td>0.57</td>
<td>0.52</td>
</tr>
<tr>
<td>20-25</td>
<td>0.51</td>
<td>0.52</td>
<td>0.50</td>
</tr>
<tr>
<td>25-30</td>
<td>0.52</td>
<td>0.52</td>
<td>0.53</td>
</tr>
<tr>
<td>30-35</td>
<td>0.52</td>
<td>0.51</td>
<td>0.52</td>
</tr>
<tr>
<td>35-40</td>
<td>0.53</td>
<td>0.52</td>
<td>0.55</td>
</tr>
<tr>
<td>40-45</td>
<td>0.53</td>
<td>0.52</td>
<td>0.53</td>
</tr>
<tr>
<td>45-50</td>
<td>0.54</td>
<td>0.54</td>
<td>0.53</td>
</tr>
<tr>
<td>50-55</td>
<td>0.53</td>
<td>0.53</td>
<td>0.52</td>
</tr>
<tr>
<td>55-60</td>
<td>0.54</td>
<td>0.54</td>
<td>0.54</td>
</tr>
<tr>
<td>60-65</td>
<td>0.53</td>
<td>0.53</td>
<td>0.54</td>
</tr>
<tr>
<td>65-70</td>
<td>0.53</td>
<td>0.52</td>
<td>0.54</td>
</tr>
<tr>
<td>70-75</td>
<td>0.52</td>
<td>0.51</td>
<td>0.53</td>
</tr>
<tr>
<td>75-80</td>
<td>0.50</td>
<td>0.50</td>
<td>0.51</td>
</tr>
<tr>
<td>80-85</td>
<td>0.48</td>
<td>0.47</td>
<td>0.48</td>
</tr>
</tbody>
</table>
Table 10
Fraction of Last Age Interval of Life, $a_i$
Ireland, 1966

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.13</td>
<td>.12</td>
<td>.13</td>
</tr>
<tr>
<td>1-5</td>
<td>.38</td>
<td>.39</td>
<td>.37</td>
</tr>
<tr>
<td>5-10</td>
<td>.47</td>
<td>.47</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.48</td>
<td>.48</td>
<td>.46</td>
</tr>
<tr>
<td>15-20</td>
<td>.55</td>
<td>.56</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.52</td>
<td>.51</td>
</tr>
<tr>
<td>35-40</td>
<td>.55</td>
<td>.56</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
<td>.55</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.50</td>
<td>.50</td>
<td>.50</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>60-65</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.49</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>80-85</td>
<td>.48</td>
<td>.48</td>
<td>.48</td>
</tr>
<tr>
<td>85-90</td>
<td>.45</td>
<td>.44</td>
<td>.46</td>
</tr>
<tr>
<td>90-95</td>
<td>.39</td>
<td>.38</td>
<td>.40</td>
</tr>
</tbody>
</table>
### Table 11

Fraction of Last Age Interval of Life, $a_i$

North Ireland, 1966

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.13</td>
<td>.13</td>
<td>.14</td>
</tr>
<tr>
<td>1-5</td>
<td>.36</td>
<td>.38</td>
<td>.35</td>
</tr>
<tr>
<td>5-10</td>
<td>.45</td>
<td>.47</td>
<td>.41</td>
</tr>
<tr>
<td>10-15</td>
<td>.50</td>
<td>.49</td>
<td>.52</td>
</tr>
<tr>
<td>15-20</td>
<td>.58</td>
<td>.59</td>
<td>.56</td>
</tr>
<tr>
<td>20-25</td>
<td>.52</td>
<td>.54</td>
<td>.48</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.53</td>
<td>.49</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.50</td>
<td>.56</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.51</td>
<td>.55</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.56</td>
<td>.57</td>
<td>.55</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>55-60</td>
<td>.55</td>
<td>.54</td>
<td>.55</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.50</td>
<td>.49</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.50</td>
<td>.49</td>
<td>.51</td>
</tr>
</tbody>
</table>
Table 12

Fraction of Last Age Interval of Life, $a_i$

Italy, 1966

<table>
<thead>
<tr>
<th>Age Interval $x_i-x_{i+1}$</th>
<th>Fraction of Last Age Interval of Life $a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Both Sexes</td>
</tr>
<tr>
<td>0-1</td>
<td>.16</td>
</tr>
<tr>
<td>1-5</td>
<td>.35</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.53</td>
</tr>
<tr>
<td>15-20</td>
<td>.53</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
</tr>
</tbody>
</table>
Table 13

Fraction of Last Age Interval of Life, $a_i$

The Netherlands, 1968

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.11</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>1-5</td>
<td>.41</td>
<td>.43</td>
<td>.39</td>
</tr>
<tr>
<td>5-10</td>
<td>.47</td>
<td>.47</td>
<td>.45</td>
</tr>
<tr>
<td>10-15</td>
<td>.51</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>15-20</td>
<td>.54</td>
<td>.55</td>
<td>.52</td>
</tr>
<tr>
<td>20-25</td>
<td>.49</td>
<td>.48</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>30-35</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.55</td>
<td>.55</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>85-90</td>
<td>.46</td>
<td>.46</td>
<td>.47</td>
</tr>
<tr>
<td>90-95</td>
<td>.42</td>
<td>.42</td>
<td>.42</td>
</tr>
</tbody>
</table>
Table 14
Fraction of Last Age Interval of Life, $a_i$
Norway, 1968

<table>
<thead>
<tr>
<th>Age Interval $x_i-x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.12</td>
<td>.10</td>
<td>.14</td>
</tr>
<tr>
<td>1-5</td>
<td>.44</td>
<td>.46</td>
<td>.42</td>
</tr>
<tr>
<td>5-10</td>
<td>.45</td>
<td>.46</td>
<td>.42</td>
</tr>
<tr>
<td>10-15</td>
<td>.56</td>
<td>.55</td>
<td>.60</td>
</tr>
<tr>
<td>15-20</td>
<td>.55</td>
<td>.56</td>
<td>.52</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>25-30</td>
<td>.48</td>
<td>.48</td>
<td>.50</td>
</tr>
<tr>
<td>30-35</td>
<td>.54</td>
<td>.55</td>
<td>.55</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.55</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.56</td>
<td>.56</td>
<td>.56</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.53</td>
<td>.52</td>
<td>.55</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>80-85</td>
<td>.50</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>85-90</td>
<td>.47</td>
<td>.46</td>
<td>.47</td>
</tr>
<tr>
<td>90-95</td>
<td>.42</td>
<td>.41</td>
<td>.43</td>
</tr>
</tbody>
</table>
Table 15
Fraction of Last Age Interval of Life, $a_i$

Okinawa, 1960

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0.32</td>
<td>0.32</td>
<td>0.31</td>
</tr>
<tr>
<td>1-5</td>
<td>0.38</td>
<td>0.37</td>
<td>0.40</td>
</tr>
<tr>
<td>5-10</td>
<td>0.45</td>
<td>0.47</td>
<td>0.45</td>
</tr>
<tr>
<td>10-15</td>
<td>0.50</td>
<td>0.51</td>
<td>0.48</td>
</tr>
<tr>
<td>15-20</td>
<td>0.50</td>
<td>0.51</td>
<td>0.49</td>
</tr>
<tr>
<td>20-25</td>
<td>0.51</td>
<td>0.53</td>
<td>0.49</td>
</tr>
<tr>
<td>25-30</td>
<td>0.52</td>
<td>0.51</td>
<td>0.53</td>
</tr>
<tr>
<td>30-35</td>
<td>0.53</td>
<td>0.52</td>
<td>0.50</td>
</tr>
<tr>
<td>35-40</td>
<td>0.50</td>
<td>0.48</td>
<td>0.52</td>
</tr>
<tr>
<td>40-45</td>
<td>0.52</td>
<td>0.51</td>
<td>0.52</td>
</tr>
<tr>
<td>45-50</td>
<td>0.53</td>
<td>0.53</td>
<td>0.54</td>
</tr>
<tr>
<td>50-55</td>
<td>0.52</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>55-60</td>
<td>0.52</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>60-65</td>
<td>0.53</td>
<td>0.52</td>
<td>0.54</td>
</tr>
<tr>
<td>65-70</td>
<td>0.53</td>
<td>0.53</td>
<td>0.53</td>
</tr>
<tr>
<td>70-75</td>
<td>0.52</td>
<td>0.52</td>
<td>0.53</td>
</tr>
<tr>
<td>75-80</td>
<td>0.52</td>
<td>0.52</td>
<td>0.52</td>
</tr>
<tr>
<td>80-85</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>Age Interval $x_i - x_{i+1}$</td>
<td>Fraction of Last Age Interval of Life, $a_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------------------------</td>
<td>---------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Both Sexes</td>
<td>Male</td>
<td>Female</td>
</tr>
<tr>
<td>0-1</td>
<td>.23</td>
<td>.23</td>
<td>.24</td>
</tr>
<tr>
<td>1-5</td>
<td>.33</td>
<td>.33</td>
<td>.33</td>
</tr>
<tr>
<td>5-10</td>
<td>.44</td>
<td>.44</td>
<td>.44</td>
</tr>
<tr>
<td>10-15</td>
<td>.49</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>15-20</td>
<td>.54</td>
<td>.53</td>
<td>.56</td>
</tr>
<tr>
<td>20-25</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>25-30</td>
<td>.49</td>
<td>.49</td>
<td>.49</td>
</tr>
<tr>
<td>30-35</td>
<td>.48</td>
<td>.48</td>
<td>.49</td>
</tr>
<tr>
<td>35-40</td>
<td>.48</td>
<td>.47</td>
<td>.50</td>
</tr>
<tr>
<td>40-45</td>
<td>.49</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>45-50</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>60-65</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.44</td>
<td>.44</td>
<td>.45</td>
</tr>
</tbody>
</table>
Table 17
Fraction of Last Age Interval of Life, $a_i$
Portugal, 1960

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i-x_{i+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>.26</td>
<td>.25</td>
<td>.27</td>
</tr>
<tr>
<td>1-5</td>
<td>.27</td>
<td>.27</td>
<td>.27</td>
</tr>
<tr>
<td>5-10</td>
<td>.42</td>
<td>.44</td>
<td>.41</td>
</tr>
<tr>
<td>10-15</td>
<td>.50</td>
<td>.50</td>
<td>.50</td>
</tr>
<tr>
<td>15-20</td>
<td>.53</td>
<td>.54</td>
<td>.52</td>
</tr>
<tr>
<td>20-25</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>70-75</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>75-80</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>80-85</td>
<td>.48</td>
<td>.47</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.45</td>
<td>.44</td>
<td>.46</td>
</tr>
<tr>
<td>90-95</td>
<td>.39</td>
<td>.38</td>
<td>.40</td>
</tr>
<tr>
<td>Age Interval</td>
<td>Fraction of Last Age Interval of Life, $a_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------------</td>
<td>------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_i$ to $x_{i+1}$</td>
<td>Both</td>
<td>Male</td>
<td>Female</td>
</tr>
<tr>
<td>0-1</td>
<td>.23</td>
<td>.22</td>
<td>.24</td>
</tr>
<tr>
<td>1-5</td>
<td>.33</td>
<td>.34</td>
<td>.32</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
<td>.47</td>
<td>.43</td>
</tr>
<tr>
<td>10-15</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>15-20</td>
<td>.56</td>
<td>.56</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.51</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.51</td>
<td>.51</td>
<td>.50</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.55</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.52</td>
<td>.55</td>
</tr>
<tr>
<td>70-75</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>Age Interval</td>
<td>Both Sexes</td>
<td>Male</td>
<td>Female</td>
</tr>
<tr>
<td>--------------</td>
<td>------------</td>
<td>------</td>
<td>--------</td>
</tr>
<tr>
<td>x_i-x_{i+1}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>.13</td>
<td>.13</td>
<td>.23</td>
</tr>
<tr>
<td>1-5</td>
<td>.40</td>
<td>.42</td>
<td>.38</td>
</tr>
<tr>
<td>5-10</td>
<td>.44</td>
<td>.44</td>
<td>.43</td>
</tr>
<tr>
<td>10-15</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>15-20</td>
<td>.56</td>
<td>.57</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.49</td>
<td>.48</td>
<td>.52</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.55</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.54</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>.50</td>
<td>.49</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
<td>.47</td>
<td>.50</td>
</tr>
</tbody>
</table>
Table 20

Fraction of Last Age Interval of Life, \( a_i \)

Spain, 1965

<table>
<thead>
<tr>
<th>Age Interval ( x_i-x_{i+1} )</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.38</td>
<td>.39</td>
<td>.37</td>
</tr>
<tr>
<td>1-5</td>
<td>.46</td>
<td>.47</td>
<td>.46</td>
</tr>
<tr>
<td>5-10</td>
<td>.53</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>10-15</td>
<td>.55</td>
<td>.56</td>
<td>.53</td>
</tr>
<tr>
<td>15-20</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.50</td>
<td>.52</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>35-40</td>
<td>.54</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>60-65</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>70-75</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>75-80</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
</tbody>
</table>
Table 21
Fraction of Last Age Interval of Life, $a_i$
Sri Lanka, 1952

<table>
<thead>
<tr>
<th>Age Interval $x_i$-$x_{i+1}$</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1*</td>
<td>.28</td>
<td>.35</td>
</tr>
<tr>
<td>1-5</td>
<td>.46</td>
<td>.45</td>
</tr>
<tr>
<td>5-10</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>10-15</td>
<td>.55</td>
<td>.42</td>
</tr>
<tr>
<td>15-20</td>
<td>.49</td>
<td>.55</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.54</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>65-70</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>75-80</td>
<td>.50</td>
<td>.45</td>
</tr>
<tr>
<td>80-85</td>
<td>.42</td>
<td>.35</td>
</tr>
<tr>
<td>85-90</td>
<td>.35</td>
<td></td>
</tr>
</tbody>
</table>

* $a_0$ values are estimated from the experience of the India 1941-50 populations
### Table 22

Fraction of Last Age Interval of Life, $a_i$

Sweden, 1966

<table>
<thead>
<tr>
<th>Age Interval $x_i - x_{i+1}$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.08</td>
<td>.08</td>
<td>.08</td>
</tr>
<tr>
<td>1-5</td>
<td>.44</td>
<td>.44</td>
<td>.45</td>
</tr>
<tr>
<td>5-10</td>
<td>.45</td>
<td>.44</td>
<td>.48</td>
</tr>
<tr>
<td>10-15</td>
<td>.53</td>
<td>.52</td>
<td>.55</td>
</tr>
<tr>
<td>15-20</td>
<td>.56</td>
<td>.57</td>
<td>.53</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.50</td>
<td>.53</td>
</tr>
<tr>
<td>25-30</td>
<td>.52</td>
<td>.53</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>.53</td>
<td>.52</td>
<td>.55</td>
</tr>
<tr>
<td>35-40</td>
<td>.52</td>
<td>.53</td>
<td>.51</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.55</td>
<td>.53</td>
</tr>
<tr>
<td>50-55</td>
<td>.54</td>
<td>.55</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>70-75</td>
<td>.53</td>
<td>.52</td>
<td>.54</td>
</tr>
<tr>
<td>75-80</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>80-85</td>
<td>.50</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>85-90</td>
<td>.46</td>
<td>.45</td>
<td>.47</td>
</tr>
<tr>
<td>90-95</td>
<td>.42</td>
<td>.41</td>
<td>.42</td>
</tr>
</tbody>
</table>
Table 23
Fraction of Last Age Interval of Life, $a_i$
Switzerland, 1968

<table>
<thead>
<tr>
<th>Age Interval</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i - x_{i+1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-1</td>
<td>10</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>1-5</td>
<td>36</td>
<td>37</td>
<td>36</td>
</tr>
<tr>
<td>5-10</td>
<td>45</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>10-15</td>
<td>52</td>
<td>54</td>
<td>47</td>
</tr>
<tr>
<td>15-20</td>
<td>57</td>
<td>58</td>
<td>52</td>
</tr>
<tr>
<td>20-25</td>
<td>49</td>
<td>48</td>
<td>49</td>
</tr>
<tr>
<td>25-30</td>
<td>49</td>
<td>50</td>
<td>48</td>
</tr>
<tr>
<td>30-35</td>
<td>51</td>
<td>53</td>
<td>49</td>
</tr>
<tr>
<td>35-40</td>
<td>54</td>
<td>54</td>
<td>53</td>
</tr>
<tr>
<td>40-45</td>
<td>53</td>
<td>53</td>
<td>54</td>
</tr>
<tr>
<td>45-50</td>
<td>55</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>50-55</td>
<td>54</td>
<td>54</td>
<td>53</td>
</tr>
<tr>
<td>55-60</td>
<td>54</td>
<td>55</td>
<td>53</td>
</tr>
<tr>
<td>60-65</td>
<td>54</td>
<td>53</td>
<td>54</td>
</tr>
<tr>
<td>65-70</td>
<td>53</td>
<td>53</td>
<td>54</td>
</tr>
<tr>
<td>70-75</td>
<td>52</td>
<td>52</td>
<td>53</td>
</tr>
<tr>
<td>75-80</td>
<td>51</td>
<td>50</td>
<td>52</td>
</tr>
<tr>
<td>80-85</td>
<td>50</td>
<td>49</td>
<td>51</td>
</tr>
<tr>
<td>85-90</td>
<td>47</td>
<td>45</td>
<td>48</td>
</tr>
<tr>
<td>90-95</td>
<td>41</td>
<td>39</td>
<td>42</td>
</tr>
</tbody>
</table>
Table 24
Fraction of Last Age Interval of Life, $a_i$
United States, 1970

<table>
<thead>
<tr>
<th>Age Interval $x_i-x_{i+1}$</th>
<th>Fraction of Last Age Interval of Life $a_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Both Sexes</td>
</tr>
<tr>
<td>0-1</td>
<td>.09</td>
</tr>
<tr>
<td>1-5</td>
<td>.40</td>
</tr>
<tr>
<td>5-10</td>
<td>.46</td>
</tr>
<tr>
<td>10-15</td>
<td>.55</td>
</tr>
<tr>
<td>15-20</td>
<td>.54</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.54</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.53</td>
</tr>
<tr>
<td>55-60</td>
<td>.53</td>
</tr>
<tr>
<td>60-65</td>
<td>.52</td>
</tr>
<tr>
<td>65-70</td>
<td>.52</td>
</tr>
<tr>
<td>70-75</td>
<td>.51</td>
</tr>
<tr>
<td>75-80</td>
<td>.51</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
</tr>
</tbody>
</table>
Table 25
Fraction of Last Age Interval of Life, $a_i$
Yugoslavia, 1968

<table>
<thead>
<tr>
<th>Age Interval $x_i-x_i+1$</th>
<th>Both Sexes</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>.23</td>
<td>.22</td>
<td>.24</td>
</tr>
<tr>
<td>1-5</td>
<td>.29</td>
<td>.31</td>
<td>.28</td>
</tr>
<tr>
<td>5-10</td>
<td>.45</td>
<td>.46</td>
<td>.43</td>
</tr>
<tr>
<td>10-15</td>
<td>.51</td>
<td>.51</td>
<td>.52</td>
</tr>
<tr>
<td>15-20</td>
<td>.53</td>
<td>.54</td>
<td>.53</td>
</tr>
<tr>
<td>20-25</td>
<td>.51</td>
<td>.52</td>
<td>.50</td>
</tr>
<tr>
<td>25-30</td>
<td>.51</td>
<td>.52</td>
<td>.50</td>
</tr>
<tr>
<td>30-35</td>
<td>.52</td>
<td>.53</td>
<td>.52</td>
</tr>
<tr>
<td>35-40</td>
<td>.53</td>
<td>.53</td>
<td>.53</td>
</tr>
<tr>
<td>40-45</td>
<td>.53</td>
<td>.52</td>
<td>.53</td>
</tr>
<tr>
<td>45-50</td>
<td>.54</td>
<td>.54</td>
<td>.54</td>
</tr>
<tr>
<td>50-55</td>
<td>.52</td>
<td>.52</td>
<td>.52</td>
</tr>
<tr>
<td>55-60</td>
<td>.54</td>
<td>.54</td>
<td>.55</td>
</tr>
<tr>
<td>60-65</td>
<td>.53</td>
<td>.53</td>
<td>.54</td>
</tr>
<tr>
<td>65-70</td>
<td>.54</td>
<td>.53</td>
<td>.55</td>
</tr>
<tr>
<td>70-75</td>
<td>.52</td>
<td>.51</td>
<td>.53</td>
</tr>
<tr>
<td>75-80</td>
<td>.49</td>
<td>.49</td>
<td>.50</td>
</tr>
<tr>
<td>80-85</td>
<td>.49</td>
<td>.48</td>
<td>.49</td>
</tr>
<tr>
<td>85-90</td>
<td>.45</td>
<td>.45</td>
<td>.46</td>
</tr>
<tr>
<td>90-95</td>
<td>.38</td>
<td>.38</td>
<td>.38</td>
</tr>
</tbody>
</table>
APPENDIX VI-A

COMPUTER PROGRAM FOR ABRIDGED LIFE TABLE CONSTRUCTION

Identification

<table>
<thead>
<tr>
<th>Program name</th>
<th>ABRIDGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author</td>
<td>Patrick Wong</td>
</tr>
<tr>
<td></td>
<td>Based on original work by Linda Kwok. Program was further modified by Carol Langhauser to handle WHO data (1969-70) in August, 1974</td>
</tr>
<tr>
<td>Department</td>
<td>Biostatistics Program</td>
</tr>
<tr>
<td></td>
<td>School of Public Health</td>
</tr>
<tr>
<td></td>
<td>University of California</td>
</tr>
<tr>
<td></td>
<td>Berkeley, California</td>
</tr>
<tr>
<td>Date</td>
<td>February, 1973</td>
</tr>
<tr>
<td>Environment</td>
<td>Machine = CDC 6400</td>
</tr>
<tr>
<td></td>
<td>Operating System = Calidoscope (SCM) version 01.2-A</td>
</tr>
<tr>
<td></td>
<td>Coding Language = FORTRAN</td>
</tr>
</tbody>
</table>

Purpose

This program constructs abridged life tables for a series of countries based on the method developed by Chin Long Chiang.

Input card preparation

All input data are assumed to be broken down into 5 year age intervals (except the first year of life) up to age 85 with the last interval being age 85 and over as follows: 0-1, 1-5, 5-10, 10-15, 15-20, 20-25, 25-30, 30-35, 35-40, 40-45, 45-50, 50-55, 55-60, 60-65, 65-70, 70-75, 75-80, 80-85, 85+. Details are given below.

1. Fractions of year lived by those dying in the interval are punched in F3.2 format consecutively starting from column one. Columns 61-80 can be used for optional population ID.

2. Title for the population date in columns 1-80. Standard format:

{TOTAL'/MALE'/FEMALE'} 'POPULATION', (country), (year)

E.g., TOTAL POPULATION, CALIFORNIA, 1970
MALE POPULATION, CANADA, 1968

3. Midyear populations of that country in each age interval in 1018 format. Two cards are required to accommodate the data for 19 age intervals.
4. Title for the death date in columns 1-80. Standard format: same as 2, except substituting the word 'DEATHS' for 'POPULATION'.

5. Number of deaths from all causes in each age interval in 1018 format. Two cards are required.

Cards in 1-5 can be repeated for as many countries as one desires. The program is terminated if a 0 read in columns 1-3 of card 1 is greater than or equal to 0.80.

Output

For each country, the following quantities are printed out:

1. Raw input data
   \( a_i \) = fractions of year lived by those dying in each age interval
   \( P_i \) = mid-year populations in each age interval
   \( D_i \) = number of deaths in each age interval

2. Construction of abridged life table
   \( x_i^1 - x_{i+1}^1 \) = age interval
   \( P_i \)
   \( D_i \)
   \( M_i \) = age specific death rate
   \( a_i \)
   \( q_i \) = proportion dying in interval

3. The abridged life table
   \( x_i^1 - x_{i+1}^1 \) = age interval
   \( q_i \)
   \( \xi_i \) = number living at age \( x_i \) \( (\xi_0 = 100,000) \)
   \( d_i \) = number dying in interval \( (x_i^1, x_{i+1}^1) \)
   \( a_i \)
   \( L_i \) = number of years lived in interval
   \( T_i \) = total number of years lived beyond age \( x_i \)
   \( e_i \) = observed expectation of life at age \( x_i \)
Program limitations

1. The current version of the program only handles 19 age intervals, broken down as described in the section, Input card preparation.

2. The maximum population size and number of deaths in any age interval has to be less than a hundred million. However, the input data format card can easily be changed to handle larger or smaller limits.

Computational procedure

1. All input data of a country \((a_1, P_1, D_1)\) are read in.

2. The age specific death rates are computed for each age interval:

   \[ M_i = \frac{D_i}{P_i} \]

3. Proportions dying in interval:

   \[ q_i = \frac{n_i M_i}{1 + (a_i)^*M_i * n_i} \]

   where \( n_i \) = length of age interval = \( x_{i+1} - x_i \).

4. Number alive at age \( x_1 \):

   \[ l_1 = l_{i-1} - d_1 \]

   In the program, the radix \( l_0 \) is set to be 100,000 for convenience.

5. Number of life table deaths in the interval:

   \[ d_1 = \hat{l}_1 q_1 \]

   Note that \( d_1 \)'s are dependent on the radix \( l_0 \)

6. Number of years lived in interval:

   \[ L_i = n_i * (\hat{l}_{i-d_i}) + a_i * n_i * d_i \]

7. Total number of years lived beyond age \( x_i \):

   \[ T_i = L_i + L_{i+1} + \ldots + L_w \]

   \[ = T_{i+1} + L_i \]

where \( x_w \) = last age interval, i.e., 85 and over.
8. Observed expectation of life at age $x_i$:

$$\hat{e}_i = \frac{T_i}{\ell_i}$$

9. In the final age interval $x_w$,

$$d_w = \ell_w$$

$$L_w = d_w / M_w$$

$$T_w = L_w$$

$$\hat{e}_w = T_w / \ell_w$$

Reference


Program listing and sample input deck setup

Caution: modifications of certain statements in the program might be required if used on machines other than CDC 6400. The obvious modifications are:

1. The first statement in the program, the Program Statement, might not be required by other machines.

2. The syntax of the read/write statements might be slightly different for different machines.

3. The input and output unit numbers for card reader and printer are probably different in different computer installations.

4. The format statements can be changed if the input data are in a different format than what this program assumes.

5. A format of A10 is used in the program for the input and output of all data titles since a maximum of ten characters can be stored in one word on a CDC machine. A different A format width and corresponding changes in the dimensions of the title arrays are called for on machines with different word structure. For example, IBM 360/370 machines only handle four characters in a word and a format of A4 has to be used when reading in or printing out character data.
PROGRAM ABRIDGE (INPUT, OUTPUT)
C PROGRAM CARD IS REQUIRED FOR CDC 6400 RUN COMPILER
C
DIMENSION TITLE (8), TITLE 2(8)
C LARGER DIMENSIONS SHOULD BE USED FOR ARRAYS TITLE AND TITLE2 FOR
C MACHINES THAT HANDLE LESS THAN 10 CHARACTERS PER WORD
C
REAL AI(20), QI(19), E(19)
INTEGER SL(19), PI(19), CL(19), DI(19), T(20)
REAL M(9)
C
C Construction of abridged life tables
C
C This program was written and debugged by Patrick Wong in Feb., 1973
C based on the preliminary work of Linda Wong.
C
C This program was further modified by Carol Langhauser
C
C Assume all input data to be broken down into the following 19 age
C intervals - 0-1, 1-5, 5-10, 10-15, 15-20, 20-25, 25-30, 30-35, 35-40, 40-45,
C 45-50, 50-55, 55-60, 60-65, 65-70, 70-75, 75-80, 80-85, 85+
C
C Input data.
C AI( ) = Fraction of last age interval of life
C PI( ) = Mid-year population in the age interval
C DI( ) = Number of deaths in the age interval
C
C Read and print data..
KCT=0
C
C Read A(I) S with optional title in col. 61-80
500 READ 1, (AI(I), I=1,20), TITLE(1), TITLE(2)
   KCT=KCT+1
   PRINT 102, KCT
   DO 272 I=1,19
      NN=19-I
      IF(AI(NN).NE.0.) GO TO 274
   272 CONTINUE
   274 CONTINUE
   PRINT 4, (AI(I), I=1,NN)
   PRINT 99, (TITLE(I), I=1,2)
   IF((0.8-AI(I)).LE.0.) GO TO 600
C
C Read title for population data
READ 108, (TITLE(I), I=1,8)
   PRINT 109, (TITLE(I), I=1,8)
C
C Read midyear populations in each age intervals
READ 2, (PI(I), I=1,19)
   PRINT 3, (PI(I), I=1,19)
C
C Read title for death data
READ 108, (TITLE2(I), I=1,8)
   PRINT 109, (TITLE2(I), I=1,8)
C
C Read numbers of deaths in each age intervals
READ 2, (DI(I), I=1,19)
   PRINT 3, (DI(I), I=1,19)
C
CHECK DATA DECK.
   LAST=-5
   DO 300 I=1,18
      IF (AI(I).GT.0.) LAST=LAST+5
   300 CONTINUE
   LSAT=-10
   DO 301 I=1,19
      IF (PI(I).GT.0.) LSAT=LSAT+5
   301 CONTINUE
   LTSA=-10
   DO 302 I=1,19
      IF (DI(I).GT.0.) LTSA=LTSA+5
   302 CONTINUE
   IF (LAST.EQ.LSAT.AND.LSAT.EQ.LTSA) GO TO 305
   PRINT 304, JJ
   GO TO 500
305 J=LAST/5+1
   QI(I)=DI(I)/(PI(I) + (1.-AI(I))*DI(I))
   QI(J+1)=1.
   SL(I)=100000
   D(I)=SL(I)*QI(I)+0.5
   CL(I)=(SL(I)-D(I))+AI(I)*D(I)+0.5
   JLAST=J+1
   DO 306 I=1,JLAST
      F=DI(I)
      G=PI(I)
      M(I)=F/G
   306 CONTINUE
C
   N=4
   DO 307 I=2,J
      QI(I)=N*M(I)/(1.+(1.-AI(I))*M(I)*N)
      TEMP=QI(I)*100000.+.5
      ITEMP=TEMP
      TEMP=ITEMP
      QI(I)=TEMP/100000.
      SL(I)=SL(I-1)-D(I-1)
      D(I)=SL(I)*QI(I)+0.5
      CL(I)=R*(SL(I)-D(I))+AI(I)*N*D(I)+0.5
      N=5
307 CONTINUE
   SL(J+1)=SL(J)-D(J)
   D(J+1)=SL(J+1)
   CL(J+1)=SL(J+1)/M(J+1)+0.5
C
   COMPUTE E(I) AND T(I)
   T(J+2)=0
   I=J+1
   308 T(I)=T(I+1)+CL(I)
      F=T(I)
      G=SL(I)
      E(I)=F/G
      I=I-1
   IF (I.GT.0) GO TO 308
C
   PRINT 8, TITLE
PRINT 14
K=1
KK=0
DO 77 I=1,J
PRINT 7,KK,K,PI(I),DI(I),M(I),AI(I),Q(I)
KK=K
K=KK+5
IF(KK.EQ.1) K=KK+4
77 CONTINUE
PRINT 5,LAST,PI(J+1),DI(J+1),M(J+1)
PRINT 100,TITLE
PRINT 9
K=1
KK=0
DO 78 I=1,J
PRINT 6,KK,K,QI(I),SL(I),D(I),AI(I),CL(I),T(I),E(I)
KK=K
K=KK+5
IF(KK.EQ.1) K=KK+4
78 CONTINUE
PRINT 13,LAST,SL(J+1),D(J+1),CL(J+1),T(J+1),E(J+1)
GO TO 500
600 PRINT 200
C
C FORMAT STATEMENTS
C CAUTION - ALL AI0 FORMATS SHOULD BE CHANGED TO APPROPRIATE
C WIDTH FOR NON CDC MACHINES
C
C OUTPUT DATA FORMATS
102 FORMAT(1H6,/,*11XXXXX8XXXXXX16XXXXXX24XXX DATA DECK NO.*,12,* PRI
INT OUT XXXXXXXX64XXXXXXX72XXXXXX80*)
4 FORMAT (1X,2OF3.2,2A10)
99 FORMAT (1H+,59X,2A10)
109 FORMAT (1X,8A1O)
3 FORMAT (1X,10I8)
119 FORMAT ( * 11XXXXX8XXXXXX16XXXXXX24XXX END OF DATA DECK NO.*,12,
1 * XXX56XXXXXX64XXXXXX72XXXXXX80*)
304 FORMAT(1H1,1X,*INPUT DATA DECK NO.*,13,2X,*IN ERROR*)
8 FORMAT (1H1,/* CONSTRUCTION OF ABRIDGED LIFE TABLE FOR *,8A10,/)
```plaintext
200 FORMAT(81H 1XXXXXXX8XXXXXX16XXXXXX24XXX END OF ALL DATA DECKS XXXX5 16XXXXXX64XXXXXX72XXXXXX80 )

C C INPUT DATA FORMATS - CAN BE MODIFIED IF NEEDED
1 FORMAT( 20F3.2,2A10)
2 FORMAT ( 10I8)
108 FORMAT( 8A10)
STOP
END

.10.42.44.55.60.49.50.52.54.54.53.54.52.51.51.50.49   CALIF. MALE, 1970
MALE POPULATION, CALIFORNIA, 1970
173822  663481  975971  998536  930884  872256  726974  611232  575226  592330
607160  529935  451259  363840  278585  202534  134280  78528   49842

MALE DEATHS, CALIFORNIA, 1970
3574   607   462   452  1432  1996  1412  1251  1596  2486
4052   5580  7596  9222 10667 11022 11042  9255   8406

FEMALE POPULATION, CALIFORNIA, 1970
166661  638717  942146  965145  886495  868710  730640  608157  574773  616220
638743  553917  481985  406930  342220  281897  207817  132425  92849

FEMALE DEATHS, CALIFORNIA, 1970
2660   442   261   283   622   706   659    713   992  1628
2670   3368  4346  5087  6421  8127 10283  10874  14077

TOTAL POPULATION, CALIFORNIA, 1970
340483 1302198 1918117 1963681 1817379 1740966 1457614 1219389 1149999 1208550
1245903 1083852  933244  770770  620805  484431  342097  210953  142691

TOTAL DEATHS, CALIFORNIA, 1970
6234  1049   723   735   2054   2702   2071  1964   2588   4114
6722   8948  11942  14309  17088  19149  21325  20129  22483
```
APPENDIX VI-B

COMPUTER PROGRAM FOR LIFE TABLE CONSTRUCTION WHEN A PARTICULAR CAUSE OF DEATH IS ELIMINATED

Identification

Program name: SPCEL T
Author: Patrick Wong
Based on original work by Linda Kwok. Program was further modified by Carol Langhauser to handle WHO data.
Department: Biostatistics Program
School of Public Health
University of California
Berkeley, California
Date: February 25, 1973
Environment: Machine = CDC 6400
Operating System = Calidoscope (SCM) version 01.2-A
Coding Language = FORTRAN

Purpose

This program constructs abridged life tables when a specific cause is eliminated as a cause of death based on the method developed by Chin Long Chiang.

Input card preparation

All input data are assumed to be broken down into most five year age intervals as follows: 0-1, 1-5, 5-10, 10-15, 15-20, 20-25, 25-30, 30-35, 35-40, 40-45, 45-50, 50-55, 55-60, 60-65, 65-70, 70-75, 75-80, 85+. Details are given below.

1. Number of specific causes of death for the following country in columns 1-2 (maximum is 25).

2. Fractions of year lived by those dying in the interval are punched consecutively in F3.2 format beginning at column 1. Columns 61-80 can be used for optimal population identification.

3. Title for population data in columns 1-80. Standard format:

'TOTAL'/ 'MALE'/ 'FEMALE' 'POPULATION', (country), (year)

e.g., TOTAL POPULATION, CANADA, 1970
    FEMALE POPULATION, AUSTRIA, 1969
4. Midyear population of that country is each age interval in 1018 format. Two cards are required.

5. Title for death data in columns 1-80. Standard format:
   Col. 1-50 'DEATH FROM ALL CAUSES'
   Col. 51-80 same as title for population data described in 3.

   e.g., Col. 1 - DEATH FROM ALL CAUSES, Col. 51 - MALE POPULATION, CANADA, 1970.

6. Number of deaths from all causes in each age interval in 1018 format. Two cards are required.

7. Title for a specific cause of death in columns 1-80. Standard format:
   Col. 1-10 'DEATH FROM'
   Col. 20-40 (specific cause of death)
   Col. 51-80 same as title for population data described in 3.

   e.g., Col. 1 - DEATH FROM INFECTIOUS DISEASES, Col. 51 - TOTAL POPULATION, USA, 1970

8. Number of deaths from that specific cause in each age interval in 1018 format. Two cards are required.

   Cards in 7,8 are to be repeated for each specific cause of death for the number of times as specified in card 1.

   Cards 1-8 can then be repeated with data from another country. The program is terminated if the number specified in card 1 is greater than 25.

Output

For each country, the following output are produced:

1. Raw input data

   \[ r \] = number of specific causes of death

   \[ a_i \] = fractions of year lived by those dying in each age interval

   \[ P_i \] = midyear population in each age interval

   \[ D_i \] = total number of deaths in each age interval

   \[ D_{i\delta} \] = number of deaths in each age interval from a specific cause, \( \delta; \delta = 1, \ldots, r \)
2. Abridged life tables when each specific cause \(R_\delta\) is eliminated as a cause of death:

\[
x_{i+1} - x_i = \text{age interval}
\]

\[
q_{i,\delta} = \text{probability that an individual alive at } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ if cause } R_\delta \text{ is eliminated as a risk of death}
\]

\[
\ell_{i,\delta} = \text{number living at age } x_i \text{ if cause } R_\delta \text{ is eliminated as a risk of death} \quad (\ell_{0,\delta} = 100,000)
\]

\[
d_{i,\delta} = \text{number dying in interval } (x_i, x_{i+1}) \text{ if cause } R_\delta \text{ is eliminated as a risk of death}
\]

\[
a_i = \text{fraction of year lived by those dying in age interval } (x_i, x_{i+1})
\]

\[
L_{i,\delta} = \text{number of years lived in interval if } R_\delta \text{ is eliminated as a risk of death}
\]

\[
T_{i,\delta} = \text{total number of years lived beyond age } x_i \text{ if } R_\delta \text{ is eliminated as a risk of death}
\]

\[
e_{i,\delta} = \text{observed expectation of life at age } x_i \text{ if } R_\delta \text{ is eliminated as a risk of death}
\]

Program limitations

1. The current version of the program only handles 19 age intervals broken down as described in the section Input card preparation.

2. The maximum population size and number of deaths in any age interval has to be less than a hundred million. However, the input data format card can easily be changed to handle larger or smaller data fields.

3. The maximum number of specific causes of death to be specified in columns 1-2 of card 1 is currently 25. This number can also be changed to handle a larger limit.

Computational procedure

1. All input data of a country \((r, a_i, P_i, D_i, D_{i,\delta})\) are read in.

2. Compute proportions dying in interval \(q_i\)

\[
q_i = \frac{n_i \cdot D_i}{P_i + (1-a_i)n_i \cdot P_i}
\]

where \(n_i = \text{length of interval } = x_{i+1} - x_i\).
3. Compute the probabilities of dying in interval \((q_{i,t})\) when a specific cause \((R_0)\) is eliminated as a cause of death

\[ q_{i,t} = 1 - \frac{1-D_{i0}/D_i}{1-q_1} \]

4. Abridged life table for cause \(R_0\) is then constructed using \(q_{i,t}\) instead of \(q_1\) following the same procedure as described in the program writeup for program ABRIDGE.

Reference


Program listing and sample deck setup

Caution: See same section in program writeup for ABRIDGE.
PROGRAM SPCZLT(INPUT, OUTPUT)
C PROGRAM CARD IS REQUIRED FOR CDC 6400 RUN COMPILER
DIMENSION AI(20), PI(19), DI(19), DC(19,25)
INTEGER PI(19), DI(19), DC(19,25)

C CONSTRUCTION OF ABRIDGED LIFE TABLE WHEN A SPECIFIC CAUSE
C IS ELIMINATED AS A CAUSE OF DEATH..
C THIS PROGRAM WAS WRITTEN AND DEBUGGED BY PATRICK WONG
C BASED ON THE PRELIMINARY WORK OF LINDA WONG.
C
C THIS PROGRAM WAS FURTHER MODIFIED BY CAROL LANGHAUSER TO HANDLE
C W.H.O. DATA
C
C INPUT DATA..
C ASSUME ALL INPUT DATA TO BE BROKEN DOWN INTO 19 AGE INTERVALS -
C LT 1, 1-5, 5-10, 10-15, 15-20, 20-25, 25-30, 30-35, 35-40, 40-45, 45-50,
C 50-55, 55-60, 60-65, 65-70, 70-75, 75-80, 80-85, 85+
C JCAUSE=NUMBER OF SPECIFIC CAUSES OF DEATH IN THE DATA DECK CONCERNED.
C MAX.=25
C AI( )=FRACTIONS OF LAST AGE INTERVAL OF LIFE.
C PI( )=MID-YEAR POPULATION IN THE AGE INTERVAL.
C DI( )=DEATH BY ALL CAUSES IN THE AGE INTERVAL
C DC( )=DEATH BY A SPECIFIC CAUSE
C
C WORKING VARIABLES..
C QI( )=LIFE TABLE PROPORTION OF DEATHS BY ALL CAUSES
C QQ( )=LIFE TABLE PROPORTION OF DEATHS WHEN A SPECIFIC CAUSE IS
C ELIMINATED AS A CAUSE OF DEATH
C
C LARGER DIMENSIONS SHOULD BE USED FOR THE FOLLOWING TITLE ARRAYS IN
C MACHINES THAT HANDLE LESS THAN 10 CHARACTERS PER WORD
REAL B,TITLE(8),TITLE1(8,25),TITLE2(8)
K=0
C
C READ NUMBER OF SPECIFIC CAUSE OF DEATH
500 READ 13,JCAUSE
K=K+1
PRINT 102,K
PRINT 15,JCAUSE
C
C PROGRAM TERMINATES IF JCAUSE GT 25
IF((26-JCAUSE).LE.0) GO TO 600
C
C READ A(I)S WITH OPTIONAL TITLE IN COL. 61-80
READ 1,(AI(I), I=1,20),TITLE2(1),TITLE2(2)
C
C CALCULATE WORKING INDEX..
DO 202 I=1,19
NN=19-I
IF(AI(NN).NE.0.) GO TO 204
202 CONTINUE
204 NM=NN+1
MM=5*NN-5
C
PRINT 4,(AI(I), I=1,NN)
PRINT 99,(TITLE2(I), I=1,2)
C
C READ TITLE FOR POPULATION DATA
READ 108,(TITLE(I), I=1,8)
PRINT 109,(TITLE(I), I=1,8)

C READ MIDYEAR POPULATIONS IN EACH AGE INTERVAL
READ 2,(PI(I), I=1,19)
PRINT3,(PI(I), I=1,19)

C READ TITLE FOR DEATH DATA
READ 108,(TITLE2(I), I=1,8)
PRINT 109,(TITLE2(I), I=1,8)

C READ NUMBERS OF DEATHS IN EACH AGE INTERVAL
READ 2,(DI(I), I=1,19)
PRINT 3,(DI(I), I=1,19)

C DO 170 J=1,JCAUSE

C READ TITLE FOR A SPECIFIC CAUSE OF DEATH
READ 108,BJTITLE1(I,J), I=1,7)
PRINT 109, B,(TITLE1(I,J), I=1,7)

C READ NUMBERS OF DEATHS FROM THAT SPECIFIC CAUSE OF DEATH IN EACH AGE INTERVAL
READ 2,(DC(I,J), I=1,19)
PRINT 3, (DC(I,J), I=1,19)
170 CONTINUE
PRINT 119,K

C COMPUTE QI( )..
DO 112 I=1,NN
N=5
IF(I.EQ.1) N=1
IF(I.EQ.2) N=4
QI(I)=N*DI(I)/(PI(I)+(1.-AI(I))*NDI(I))
112 CONTINUE

C COMPUTE THE PROBABILITY OF DYING QQ ( ) WHEN A SPECIFIC CAUSE IS ELIMINATED AS A CAUSE OF DEATH
DO 700 J=1,JCAUSE
DO 110 I=1,NN
F=DC(I,J)
G=DI(I)
EE=1.-F/G
QQ(I)=1.-((1.-QI(I))**EE
TEMP=QQ(I)*100000.+0.5
ITEMP = TEMP
TEMP=ITEMP
QQ(I)=TEMP/100000.
110 CONTINUE
QQ(NM)=1.

C F=DI(NM)-DC(NM,J)
G=PI(NM)
WM=F/G
PRINT 7,(TITLE(I), I=1,5)
PRINT 10,(TITLE1(I,J), I=1,3)
PRINT 9
CALL ABRLIF(AI,QQ,WM,NN)
700 CONTINUE
GO TO 500
600 PRINT 100
C FORMAT STATEMENTS
C CAUTION - ALL A10 FORMATS SHOULD BE CHANGED TO APPROPRIATE WIDTH
C FOR NON CDC MACHINES
C
C INPUT DATA FORMATS - CAN BE CHANGED IF NEEDED
1 FORMAT( 20F3.2,2A10)
2 FORMAT( 10I8)
13 FORMAT(I2)
108 FORMAT( 8A10)
C
C OUTPUT DATA FORMATS
C
3 FORMAT(1X,10I8)
4 FORMAT(1X,20F3.2,2A10)
7 FORMAT( 1H1,,/25H ABRIDGED LIFE TABLE FOR ,5A10)
9 FORMAT(/3X,*AGE*,12X,*PROPORTION*,4X,*NUMBER*,2X,*NUMBER*,8X,*FRAC
TION*,5X,*NUMBER*,9X,*TOTAL*,11X,*OBSERVED*,3X,*INTERVAL*,1X,*DY
RING IN*,6X,*LIVING*,2X,*DYING IN *,6X,*OF LAST*,6X,*OF YEARS*,7X,*N
UMBER OF*,7X,*EXPECTATION*/3X,*IN YEARS*),5X,*INTERVAL*,6X,*AT AG
4E*,2X,*INTERVAL*,6X,*AGE INTERVAL*,1X,*LIVED IN*,7X,*YEARS LIVED*,
55X,*OF LIFE AT*/3X,*X(I) TO X(I+1)*,1X,*X(I),X(I+1))*,1X,*X(I)*,4
6X,*X(I),X(I++1)),1X,*OF LIFE*,6X,*INTERVAL*,7X,*BEYOND AGE X(I)*,
71X,*AGE X(I)*/67X,*X(I),X(I+1))/,20X,*Q(1.1)*6X,*SL(1)*,7X,*D(I)
8*, 8X,*A(1)*,9X,*CL(1)*,11X,*T(1)*,11X,*E(1.1))/
10 FORMAT(5H WHEN 1X,R9,2A10/35H IS ELIMINATED AS A CAUSE OF DEATH )
15 FORMAT(1X, 12)
99 FORMAT(1H+,59X,2A10)
100 FORMAT (81H 1 XXXXXXXXX16XXXXXX24XXX END OF ALL DATA DECKS XXXX5
16XXXXXX64XXXXXX72XXXXXX80 )
102 FORMAT(1H7,/*11XXXXXX8XXXXXX16XXXXXX24XXX DATA DECK NO.*,12,* PRI
1NT OUT XXXXXXXXX64XXXXXX72XXXXXX80*)
109 FORMAT(1X,8A10)
119 FORMAT( * 1XXXXXX8XXXXXX16XXXXXX24XXX END OF DATA DECK No.*,12,
1 * XXX56XXXXXX64XXXXXX72XXXXXX80*)
STOP
END
SUBROUTINE ABRLIF(AI,QI,WM,J)
C CONSTRUCTION OF ABRIDGED LIFE TABLE..
DIMENSION AI(20),QI(19),E(19)
INTEGER SL(19),CL(19),D(19),T(20)
C
C COMPUTE D( ),SL( ),CL( )..
SL(1)=100000
D(1)=SL(1)*QI(1)+0.5
CL(1)=(SL(1)-D(1))+AI(1)*D(1)+0.5
N=4
DO 307 I=2, J
SL(I)=SL(I-1)-D(I-1)
D(I)=SL(I)*QI(I)+0.5
CL(I)=N*(SL(I)-D(I))+AI(I)*N*D(I)+0.5
N=5
307 CONTINUE
SL(J+1)=SL(J)-D(J)
D(J+1)=SL(J+1)
CL(J+1)=SL(J+1)/WM+0.5
C
C COMPUTE E( ) AND T( )..
T(J+2)=0
\[ I = J + 1 \]
\[ T(I) = T(I+1) + CL(I) \]
\[ F = T(I) \]
\[ G = SL(I) \]
\[ E(I) = F/G \]
\[ 1 = 1 - 1 \]
\[ IF(I.GT.0) \text{ GO TO 308} \]
\[ K = 1 \]
\[ KK = 0 \]
\[ DO 78 I = 1, J \]
\[ PRINT 6, KK, K, QI(I), SL(I), D(I), AI(I), CL(I), T(I), E(I) \]
\[ KK = K \]
\[ K = KK + 5 \]
\[ IF(KK.EQ.1) \text{ K = KK + 4} \]
\[ 78 \text{ CONTINUE} \]
\[ \text{LAST} = (J-1)*5 \]
\[ PRINT 13, \text{LAST}, SL(J+1), D(J+1), CL(J+1), T(J+1), E(J+1) \]

**DEATHS FROM ALL CAUSES**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 1**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 2**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 3**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 4**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 5**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 6**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 7**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 8**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 9**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
</tbody>
</table>

**DEATHS FROM CAUSE 10**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Total Population, Canada, 1970</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1276900 1295100 1297800 1308500 1317000 1325300 1332800 1340300 1347800 1355300</td>
</tr>
<tr>
<td>Cause 11</td>
<td>Cause 12</td>
</tr>
<tr>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>Deaths</td>
<td>Deaths</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>168</td>
<td>0</td>
</tr>
<tr>
<td>342</td>
<td>0</td>
</tr>
<tr>
<td>260</td>
<td>260</td>
</tr>
<tr>
<td>881</td>
<td>143</td>
</tr>
<tr>
<td>204</td>
<td>204</td>
</tr>
<tr>
<td>186</td>
<td>186</td>
</tr>
<tr>
<td>143</td>
<td>143</td>
</tr>
<tr>
<td>124</td>
<td>124</td>
</tr>
<tr>
<td>85</td>
<td>85</td>
</tr>
<tr>
<td>62</td>
<td>62</td>
</tr>
<tr>
<td>258</td>
<td>258</td>
</tr>
<tr>
<td>221</td>
<td>221</td>
</tr>
<tr>
<td>211</td>
<td>211</td>
</tr>
<tr>
<td>150</td>
<td>150</td>
</tr>
<tr>
<td>117</td>
<td>117</td>
</tr>
<tr>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>42</td>
<td>42</td>
</tr>
<tr>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>
REFERENCES


Chiang, C. L. [1961a]. A stochastic study of the life table and its applications: III. The follow-up study with the consideration of competing risks, 17, 57-78.


the National Academy of Science, 25, 461-467.

Doll, R. and P. Cook [1967]. Summarizing indices for comparison of cancer

Dorn, H. [1950]. Methods of analysis for follow-up studies. Human Biol., 22,
238-248.

135-144.

Journal of the American Public Health Assoc.

Dublin, L. I., A. J. Lotka, and M. Spiegelman [1949]. Length of Life: A Study
of the Life Table. Ronald Press, New York.

Duda, R. and G. Duda [1967]. Life tables of the population of the city of


and surveys. Bulletin of the International Statistical Institute 36, Part 2,
12-35. Stockholm.

Efron, B. [1965]. The two sample problem with censored data. Proc. 5th

A life table approach to analysis of family data. J. Chronic Dis., 26,
529-45.

El-Badry, M. A. [1969]. Higher female than male mortality in some countries
of South Asia: A digest. J. Amer. Statist. Assoc., 64, 1234-1244.

Elveback, L. [1958]. Estimation of survivorship in chronic disease: The

Elveback, L. R. [1966]. Discussion of "Indices of mortality and tests of

486-502.

Fabia, J. and M. Drolette [1970]. Life tables up to age 10 for mongols with


Mantel, N. and W. Haenszel [1959]. Statistical aspects of the analysis of data from retrospective studies of disease. Journal of the National Cancer Institute, 22.


GLOSSARY
CHAPTER 1
Formulas

Probability of A:

$$\Pr(A) = \frac{n(A)}{n} \tag{2.1}$$

$$0 \leq \Pr(A) \leq 1 \tag{2.2}$$

$$\Pr(\overline{A}) = 1 - \Pr(A) \tag{2.5a}$$

$$\Pr(AB) = \frac{n(AB)}{n} \tag{2.6}$$

Conditional probability:

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{2.9}$$

Multiplication theorem:

$$\Pr(AB) = \Pr(A) \times \Pr(B|A) \tag{2.13}$$

$$\Pr(ABC) = \Pr(A) \times \Pr(B|A) \times \Pr(C|AB) \tag{2.15}$$

$$\Pr(ABCD) = \Pr(A) \times \Pr(B|A) \times \Pr(C|AB) \times \Pr(D|ABC) \tag{2.16}$$

Addition theorem:

$$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.21}$$

$$\Pr(A \text{ or } B \text{ or } C) = \Pr(A) + \Pr(B) + \Pr(C)$$

$$- \Pr(AB) - \Pr(BC) - \Pr(CA) + \Pr(ABC) \tag{2.22}$$

$$\Pr(A \text{ or } B \text{ or } C \text{ or } D) = \Pr(A) + \Pr(B) + \Pr(C) + \Pr(D)$$

$$- \Pr(AB) - \Pr(AC) - \Pr(AD) - \Pr(BC) - \Pr(BD) - \Pr(CD)$$

$$+ \Pr(ABC) + \Pr(ABD) + \Pr(ACD) + \Pr(BCD) - \Pr(ABCD) \tag{2.23}$$
Distributive law:

Pr\(A(B \text{ or } C)) = Pr(AB \text{ or } AC) \tag{2.27}

Pr\((A \text{ or } B)(C \text{ or } D)) = Pr(AC \text{ or } AD \text{ or } BC \text{ or } BD) \tag{2.28}


CHAPTER 2

\( a_i \) - Fraction of the last age interval of life. It is the expected fraction of the interval \((x_i, x_{i+1})\) lived by an individual who dies at an age in the interval \((x_i, x_{i+1})\).

\( d_i \) - Number of life table deaths in the age interval \((x_i, x_{i+1})\).

\( D \) - Total number of deaths in a current population.

\( D_{i0} \) - Number of deaths from cause \( R_{i0} \) in a current population.

\( D_i \) - Number of deaths in the age group \((x_i, x_{i+1})\) in a current population.

\( D_{i0i} \) - Number of deaths from cause \( R_{i0} \) in age group \((x_i, x_{i+1})\) in a current population.

\( D_s \) - Total number of deaths in a standard population.

\( D_{si} \) - Number of deaths in the age interval \((x_i, x_{i+1})\) in the standard population.

\( D_u \) - Total number of deaths in community \( u \).

\( D_{ui} \) - Number of deaths in the age interval \((x_i, x_{i+1})\) in community \( u \).

\( e_0 \) - Observed expectation of life at age zero.

\( l_i \) - Number alive at exact age \( x_i \) in the life table population.

\( L_{i} \) - Number of years lived in \((x_i, x_{i+1})\) by \( l_i \) individuals.

\( n_i \) - Length of the age interval \((x_i, x_{i+1})\); \( n_i = x_{i+1} - x_i \).

\( N_i \) - (Hypothetical) number of individuals alive at exact age \( x_i \).

\( P \) - Total midyear population.

\( P_{i} \) - Midyear population in age interval \((x_i, x_{i+1})\).

\( P_s \) - Total midyear standard population.

\( P_{si} \) - Midyear population in the age interval \((x_i, x_{i+1})\) of the standard population.

\( P_u \) - Total midyear population of community \( u \).

\( P_{ui} \) - Midyear population in age interval \((x_i, x_{i+1})\) of community \( u \).
\[ T_0 \] - Total number of years lived by the life table population beyond \( x_0 \).

\[ x_i \] - Exact age in years at the lower limit of the \( i \)-th interval.

\[ x_{i+1} \] - Exact age in years at the upper limit of the \( i \)-th interval.

Age-specific death rate - \[ M_i = \frac{\text{Number dying in } (x_i, x_{i+1})}{\text{Number of years lived in } (x_i, x_{i+1}) \text{ by those alive at } x_i} \]

Fetal death rate - (alias "stillbirth rate). Two definitions are available:

\[
\frac{\text{Number of fetal deaths or 28 or more weeks of gestation}}{\text{Number of live births} + \text{fetal deaths of 28 or more weeks of gestation}} \times 1000
\]

\[
\frac{\text{Number of fetal deaths of 20 weeks or more of gestation}}{\text{Number of live births} + \text{fetal deaths of 20 or more weeks of gestation}} \times 1000
\]

Neonatal mortality rate - \[ \frac{\text{Number of deaths under 28 days of age}}{\text{Number of live births}} \times 1000 \]

Perinatal mortality rate - There are two definitions in common use:

\[
\frac{\text{Number of deaths under 7 days} + \text{fetal deaths of 28 weeks or more of gestation}}{\text{Number of live births} + \text{fetal deaths of 28 weeks or more of gestation}}
\]

\[
\frac{\text{Number of deaths under 28 days of life} + \text{fetal deaths of 20 or more weeks of gestation}}{\text{Number of live births} + \text{fetal deaths of 20 or more weeks of gestation}} \times 1000
\]

Post neonatal mortality rate - \[ \frac{\text{Number of deaths at age 28 days through one year}}{\text{Number of live births} - \text{neonatal deaths}} \times 1000 \]

Infant mortality rate - \[ \frac{\text{Number of deaths under one year of age}}{\text{Number of live births}} \times 1000 \]

Fetal death ratio - \[ \frac{\text{Number of fetal deaths of 20 or more weeks of gestation}}{\text{Number of live births}} \]

Maternal mortality rate - \[ \frac{\text{Number of maternal deaths}}{\text{Number of live births}} \times 1000 \]
Probability of dying for age interval \( (x_i, x_{i+1}) \):

\[
\hat{q}_i = \frac{d_i}{N_i},
\]
\[
\hat{q}_i = \frac{D_i}{N_i}.
\]  

(1.3)

(1.3a)

Age-specific death rate for age interval \( (x_i, x_{i+1}) \):

\[
M_i = \frac{d_i}{n_1(x_{i-1} - d_i) + a_i n_1 d_i},
\]
\[
M_i = \frac{D_i}{n_1(N_i - D_i) + a_i n_1 D_i},
\]
\[
M_i = \frac{D_i}{P_1}.
\]  

(1.2)

(1.2a)

(1.5)

Relationship between \( q_i \) and \( M_i \) for age interval \( (x_i, x_{i+1}) \):

\[
\hat{q}_i = \frac{n_1 M_i}{1 + (1-a_i) n_1 M_i}.
\]

(1.4)

Cause-specific death rate for cause \( R_\delta \):

\[
M_\delta = \frac{D_\delta}{P} \times 100,000.
\]  

(1.9)

Age-cause-specific death rate for cause \( R_\delta \) age interval \( (x_i, x_{i+1}) \):

\[
M_{i\delta} = \frac{D_{i\delta}}{P_1} \times 100,000.
\]  

(1.10)
Crude death rate:
\[ M = \frac{D}{P} \times 1000 \]  

Crude death rate for community u:
\[ \text{C.D.R.} = \frac{D_u}{P_u} \]  

Age-specific death rate for community u and age interval \((x_i, x_{i+1})\):
\[ M_{ui} = \frac{D_{ui}}{P_{ui}} \]  

Crude death rate (as a weighted average of \(M_{ui}\)):
\[ \text{C.D.R.} = \sum_i \frac{P_{ui}}{P_u} M_{ui} \]  

Direct method of adjustment:
\[ \text{D.M.D.R.} = \sum_i \frac{P_{si} M_{ui}}{P_s} \]  

Comparative mortality ratio:
\[ \text{C.M.R.} = \frac{1}{i} \sum_i \left( \frac{P_{ui}}{P_u} + \frac{P_{si}}{P_s} \right) M_{ui} \]  

Indirect method of adjustment:
\[ \text{I.M.D.R.} = \sum_i \frac{D_s/P_s}{P_{ui}} M_{si}/P_u \]  

Life table death rate:
\[ \text{L.T.D.R.} = \sum_i \frac{L_i}{T_i} M_{ui} \]  
\[ \text{L.T.D.R.} = \frac{1}{e_0} \]
Equivalent average death rate:

\[ \text{E.A.D.R.} = \sum_{i} \frac{n_i}{\sum_{i} n_i} M_{ui} \] (3.22)

Relative mortality index

\[ \text{R.M.I.} = \sum_{i} \frac{P_{ui}}{P_u} \frac{M_{ui}}{M_{si}} \] (3.23)

Mortality index:

\[ \text{M.I.} = \frac{\sum_{i} M_{ui}}{\sum_{i} M_{si}} \] (3.24)

Standardized mortality ratio:

\[ \text{S.M.R.} = \frac{\sum_{i} P_{ui} M_{ui}}{\sum_{i} P_{ui} M_{si}} \] (3.25)
CHAPTER 3

\( a_i \) - Fraction of the last age interval of life.

\( D_i \) - The number of deaths in the age group \((x_i, x_{i+1})\) in a current population.

\( E(D_i) \) - Expected number of deaths in the interval \((x_i, x_{i+1})\)

\( D_s \) - The total number of deaths in the standard population.

\( D_u \) - The total number of deaths in community \( u \).

\( E(q_i | N_i) \) - The conditional expectation of \( q_i \) given \( N_i \).

\( L_i \) - Number of years lived in \((x_i, x_{i+1})\) by \( i \) individuals.

\( M_i \) - The age specific death rate for interval \((x_i, x_{i+1})\)

\( M_{si} \) - The age-specific death rate for interval \((x_i, x_{i+1})\)
in the standard population.

\( M_{ui} \) - The age-specific death rate for interval \((x_i, x_{i+1})\)
in community \( u \).

\( n_i \) - The length of the age interval \((x_i, x_{i+1})\): \( n_i = x_{i+1} - x_i \).

\( N_i \) - (Hypothetical) number of individuals alive at exact age \( x_i \).

\( P_i \) - Midyear population in age interval \((x_i, x_{i+1})\).

\( P_s \) - Total midyear standard population.

\( P_{si} \) - Midyear population in the age interval \((x_i, x_{i+1})\) of the standard population.

\( P_u \) - Total midyear population of community \( u \).

\( P_{ui} \) - Midyear population in age interval \((x_i, x_{i+1})\) of community \( u \).

\( q_i \) - Probability of death in the interval \((x_i, x_{i+1})\).

\( \hat{q}_i \) - Estimate of the probability of death in the interval \((x_i, x_{i+1})\).

\( \hat{q}_{ui} \) - Estimate of the probability of death in the interval \((x_i, x_{i+1})\) of community \( u \).

\( \rho \) - General symbol for an adjusted rate or mortality index.

\( S^2 \) - The sample variance of \( D_i \).
$S_{q_i}^2$ - The sample variance of $q_i$.  

$S_{R}^2$ - Sample variance of an adjusted rate or mortality index, $R$.  

$\sigma_{D_i}^2$ - Variance of $D_i$.  

$\sigma_{q_i}^2$ - Variance of $q_i$.  

$\sigma_{q_i | N_i}^2$ - Conditional variance of $q_i$ given $N_i$.  

$w_i$ - Weight of $M_{u_i}$ used to calculate adjusted rates and mortality indices.  

$T_j$ - Total number of years lived by the life table population beyond $x_0$.  

$x_i$ - Exact age in years at the lower limit of the age interval.  

$\nu_{i+1}$ - Exact age in years at the upper limit of the age interval.  

Binomial Distribution - If an event has a constant probability $q$ of occurring in any one trial, then the number of times $(D)$ that the event will occur in $N$ independent trials has a binomial distribution, with the expected value $E(D) = Nq$ and variance $\sigma_D^2 = Nq(1-q)$.  

Coefficient of Variation of $R$ - A measure of the magnitude of the standard deviation of an adjusted rate, $R$, relative to $R$ itself.
Formulas

Expectations and variances of the number of deaths and probability of dying in \((x_i, x_{i+1})\):

\[ E(D_i) = N_i q_i \quad (2.1) \]

\[ \hat{q}_i = \frac{D_i}{N_i} \quad (2.3) \]

\[ \sigma^2_{D_i} = N_i q_i (1-q_i) \quad (2.2) \]

\[ E(\hat{q}_i) = E\left( \frac{D_i}{N_i} \right) = \frac{1}{N_i} E(D_i) = \frac{1}{N_i} N_i q_i = q_i \quad (2.4) \]

\[ \sigma^2_{\hat{q}_i} = \frac{1}{N_i} q_i (1-q_i) \quad (2.5) \]

\[ S^2_{\hat{q}_i} = \frac{1}{N_i} \hat{q}_i (1-\hat{q}_i) \quad (2.6) \]

95% confidence interval for \(q_i\):

\[ Z = \frac{\hat{q}_i - q_i}{\sqrt{\frac{\hat{q}_i (1-\hat{q}_i)}{N_i}}} \quad (2.7) \]

\[ \hat{q}_i - 1.96 S_{\hat{q}_i} < q_i < \hat{q}_i + 1.96 S_{\hat{q}_i} \quad (2.10) \]
Sample variances of $M_i$, $q_i$, and $q_x$:

\[ \sigma^2_{M_i} = \hat{\sigma}_i \hat{\sigma}_i (1-\hat{\sigma}_i) = \mu_i (1-\mu_i) \]  
\[ (3.7) \]

\[ \sigma^2_i = \frac{\sigma_i}{\mu_i} \]  
\[ (2.3) \]

\[ M_i = \frac{D_i}{p_i} \]  
\[ (3.4) \]

\[ S^2_{\hat{q}_i} = \frac{1}{D_i} \hat{q}_i^2 (1-\hat{q}_i) \]  
\[ (3.5) \]

\[ S^2_{q_x} = \frac{1}{D_x} \hat{q}_x^2 (1-\hat{q}_x) \]  
\[ (8.5) \]

\[ S^2_{M_i} = \frac{1}{p_i} M_i (1-\hat{q}_i) \]  
\[ (3.8) \]

Sample variance of age adjusted death rates:

\[ R = \sum w_i M_i u_i \]  
\[ (6.1) \]

\[ S^2_R = \sum w_i^2 S^2_{M_i u_i} \]  
\[ (6.3) \]

\[ S^2_R = \sum w_i^2 \frac{M_i u_i}{p_i u_i} (1-\hat{q}_{u_i}) \]  
\[ (6.4) \]
Sample variance of direct method of age adjustment:

\[ S^2_{DMDR} = \sum_i \left( \frac{p_i s^2_{ui}}{p_i} \right)^2 \frac{2}{u_i} \left( 1 - \hat{q}_{ui} \right) \]  

(7.1)

Coeff. of variation of \( R = \frac{S_R}{R} \)  

(7.5)

Sample variance for life table death rate:

\[ \text{LTDR} = \frac{\sum X M u_x}{\sum X} = \frac{\sum d_x}{\sum L_x} = \frac{\hat{r}_0}{T_0} = \frac{1}{\hat{e}_0} . \]  

(8.1)

\[ S^2_{\text{LTDR}} = \frac{1}{\hat{e}_0^4} S^2_{\hat{e}_0} . \]  

(8.2)

\[ S^2_{\hat{e}_0} = \frac{1}{\hat{e}_0^4} \sum_{x \geq 0} \hat{p}_0 x \left[ (1 - a_x) n_x + \hat{e}_x n_x \right]^2 S^2_{a_x} . \]  

(8.4)
Summary of adjusted death rates and indices (Table 1):

<table>
<thead>
<tr>
<th>Index</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude death rate (C.D.R.)</td>
<td>$\frac{\sum P \cdot M}{\sum i \cdot u \cdot ui / P}$</td>
</tr>
<tr>
<td>Direct method of adjustment (D.M.D.R.)</td>
<td>$\frac{\sum P \cdot M}{\sum i \cdot si \cdot ui / P}$</td>
</tr>
<tr>
<td>Comparative mortality rate (C.M.R.)</td>
<td>$\frac{1}{2} \sum \left( \frac{P \cdot M}{P \cdot u + si} \right) \cdot ui$</td>
</tr>
<tr>
<td>Indirect method of adjustment (I.M.D.R.)</td>
<td>$\frac{\sum P \cdot M \cdot (D/P) - D/s}{\sum u \cdot i \cdot si / P}$</td>
</tr>
<tr>
<td>Life table death rate (L.T.D.R.)</td>
<td>$\frac{\sum L \cdot M}{\sum i \cdot u \cdot ui}$</td>
</tr>
<tr>
<td>Equivalent average death rate (E.A.D.R.)</td>
<td>$\frac{\sum n \cdot M}{\sum i \cdot u \cdot M}$</td>
</tr>
<tr>
<td>Relative mortality index (R.M.I.)</td>
<td>$\frac{\sum P \cdot ui \cdot M}{\sum u \cdot M \cdot si}$</td>
</tr>
<tr>
<td>Mortality index (M.I.)</td>
<td>$\frac{\sum n \cdot M \cdot ui}{\sum i \cdot M \cdot si}$</td>
</tr>
<tr>
<td>Standardized mortality ratio (S.M.R.)</td>
<td>$\frac{\sum M \cdot ui}{\sum P \cdot u \cdot si}$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$a'_x$</td>
<td>Fraction of the last year of life.</td>
</tr>
<tr>
<td>$d'_x$</td>
<td>Number of life table deaths in the age interval $(x, x+1)$.</td>
</tr>
<tr>
<td>$D_x$</td>
<td>Number of deaths in the age interval $(x, x+1)$ in a current population.</td>
</tr>
<tr>
<td>$e_x$</td>
<td>The expectation of life at age $x$.</td>
</tr>
<tr>
<td>$\hat{e}_x$</td>
<td>Observed expectation of life at age $x$.</td>
</tr>
<tr>
<td>$l_x$</td>
<td>Number alive at exact age $x$ in a life table population.</td>
</tr>
<tr>
<td>$L_x$</td>
<td>Number of years lived in $(x, x+1)$ by the $l_x$ individuals.</td>
</tr>
<tr>
<td>$M_x$</td>
<td>Age-specific death rate in the interval $(x, x+1)$.</td>
</tr>
<tr>
<td>$N_x$</td>
<td>(Hypothetical) number of individuals alive at exact age $x$.</td>
</tr>
<tr>
<td>$p_x$</td>
<td>Proportion of those alive at age $x$ surviving the interval $(x, x+1)$.</td>
</tr>
<tr>
<td>$\hat{p}_{xy}$</td>
<td>Proportion of those alive at age $x$ surviving to age $y$.</td>
</tr>
<tr>
<td>$P_x$</td>
<td>Midyear population in age interval $(x, x+1)$.</td>
</tr>
<tr>
<td>$\hat{q}_x$</td>
<td>Estimate of the probability of dying in $(x, x+1)$.</td>
</tr>
</tbody>
</table>
\[ T_x \]
- Total number of years lived by the life table population beyond age \( x \).

\[ x \]
- Lower limit of age interval \((x, x+1)\).

\[ x+1 \]
- Upper limit of age interval \((x, x+1)\).

\[ x_w \]
- Lower limit of the final age interval in a life table.

Abridged Life Table
- A life table with age intervals greater than one year (beyond age 1).

Complete Life Table
- A life table with single year age intervals.

Current Life Table
- A life table based on current mortality and population data.

Cohort Life Table
- A life table based on the mortality experience of a single group of individuals.
Formulas

Relationship between life table functions in the complete life table:

\[ d_x = t_x \delta_x, \quad x=0,1,\ldots,\omega \quad (2.1) \]

\[ t_{x+1} = t_x - d_x, \quad x=0,1,\ldots,\omega-1 \quad (2.2) \]

\[ L_x = (t_x - d_x) + a_x d_x, \quad x=0,1,\ldots,\omega-1 \quad (2.3) \]

\[ T_x = L_x + L_{x+1} + \ldots + L_{\omega}, \quad x=0,1,\ldots,\omega \quad (2.5) \]

\[ T_x = L_x + T_{x+1} \quad (2.6) \]

\[ \hat{t}_x = \frac{T_x}{L_x}, \quad x=0,1,\ldots,\omega. \quad (2.7) \]

\[ \hat{p}_x = 1 - \hat{t}_x \quad (2.8) \]

\[ \hat{p}_{xy} = \hat{p}_x \hat{p}_{x+1} \cdots \hat{p}_{y-1} = \frac{\ell_y}{l_x} \quad (2.9) \]
Computation of $q_x$, $l_w$, $T_w$, and $e_w$:

\[ \hat{q}_x = \frac{D}{N} \quad . \]  
(3.1)

\[ M_x = \frac{D}{(N-D)+aD} \quad . \]  
(3.2)

\[ M_x = \frac{D}{p} \quad . \]  
(3.4)

\[ \hat{q}_x = \frac{M}{1 + (1-a)M} \quad . \]  
(3.7)

\[ L_w = \frac{w}{M} \quad . \]  
(3.11)

\[ T_w = L_w \quad \text{and} \quad \hat{e}_w = \frac{T}{\hat{e}} = \frac{L_w}{d_w} = \frac{1}{M_w} \quad . \]  
(3.12)
CHAPTER 5

Page no. 69

Fraction of the last age interval of life.

- Number of life table deaths in the age interval \((x_i, x_{i+1})\)

- Number of deaths in the age interval \((x_i, x_{i+1})\) in a current population

- Observed expectation of life at age \(x_i\).

- Number alive at exact age \(x_i\) in a life table population.

- Number of years lived in the interval \((x_i, x_{i+1})\) by the \(l_i\) individuals.

- Age-specific death rate in the interval \((x_i, x_{i+1})\).

- Length of the age interval \((x_i, x_{i+1})\); \(n_i = x_{i+1} - x_i\).

- (Hypothetical) number of individuals alive at exact age \(x_i\).

- Midyear population in age interval \((x_i, x_{i+1})\).

- Estimate of the probability of dying in interval \((x_i, x_{i+1})\).

- Total number of years lived by the life table population beyond age \(x_i\).

- Lower limit of age interval \((x_i, x_{i+1})\).

- Upper limit of age interval \((x_i, x_{i+1})\).

Formulas

Construction of abridged life table:

\[
\hat{q}_i = \frac{D_i}{N_i} \quad (2.1)
\]

\[
x_i = \frac{D_i}{(N_i - D_i) n_i + a_i n_i D_i} \quad (2.2)
\]

\[
\hat{q}_i = \frac{n_i M_i}{1 + (1-a_i)n_i M_i} \quad (2.3)
\]
\[ \mathcal{M}_i = \frac{D_i}{P_i} \quad \text{(2.4)} \]

\[ d_i = \tilde{l}_i \tilde{q}_i, \quad i=0,1,\ldots,w-1, \quad \text{(2.5)} \]

\[ \tilde{l}_{i+1} = \tilde{l}_i - d_i, \quad i=0,1,\ldots,w-1, \quad \text{(2.6)} \]

\[ L_i = n_i (\tilde{l}_i - d_i) + a_i n_i d_i, \quad i=0,1,\ldots,w-1. \quad \text{(2.7)} \]

\[ L_w = \frac{L}{\tilde{l}_w}, \quad \text{(2.8)} \]

\[ \tilde{e}_i = \frac{L_i + L_i+1 + \ldots + L_w}{\tilde{l}_i}, \quad i=0,\ldots,w. \quad \text{(2.9)} \]
Computation of the fraction of last age interval of life, \( a_1 \):

\[
a_1 = \frac{q_1^2 + p_1 q_1 (1 + a_1^2) + p_1 p_2 q_3 (2 + a_3^2) + p_1 p_2 p_3 q_4 (3 + a_4^2)}{4(1 - p_1 p_2 p_3 p_4)}. \quad (3.1)
\]

\[
a_2 = \frac{.5q_5 + (1 + .5)p_5 q_6 + (2 + .5)p_6 q_7 + (3 + .5)p_7 q_8 + (4 + .5)p_8 q_9}{5(1 - p_5 p_6 p_7 p_8 p_9)} + .1, \quad (3.3)
\]

\[
q_5 + p_5 q_6 + p_5 p_6 q_7 + p_6 p_7 q_8 + p_5 p_6 p_7 q_9 = 1 - p_5 p_6 p_7 p_8 p_9. \quad (3.4)
\]
Computation of cohort life table functions:

\[ \ell_x - \ell_{x+n} = d_x \]  
\[ \ell_{x+n} = \ell_x - d_x \]  
\[ \hat{q}_x = \frac{d_x}{\ell_x} \]  
\[ L_x = \ell_{x+n} + (1-a_x)nd_x \]  
\[ T_x = L_x + \ldots + L_w \]  
\[ e_x = \frac{T_x}{L_x}, \quad x = 0,1,\ldots,w \]
CHAPTER 6

$a_i$ - Fraction of the last age interval of life.

$d_i$ - Number of life table deaths in the age interval $(x_i, x_{i+1})$.

$D_i$ - Number of deaths in the age interval $(x_i, x_{i+1})$ in a current population.

$e_\alpha$ - Expectation of life at age $x_\alpha$.

$\hat{e}_\alpha$ - Observed expectation of life at age $x_\alpha$.

$L_0$ - Life table population at age $x_0$. It is an arbitrarily assigned number and is referred to as the radix.

$L_i$ - Number alive at exact age $x_i$ in the life table population.

$L_n$ - Number alive at exact age $x_n$.

$L_{i+1}$ - Number of years lived in the interval $(x_i, x_{i+1})$ by the $L_i$ individuals.

$n_i$ - Length of the age interval $(x_i, x_{i+1})$; $n_i = x_{i+1} - x_i$.

$p_i$ - Probability of surviving the interval $(x_i, x_{i+1})$.

$\hat{p}_i$ - Estimate of the probability of surviving the interval $(x_i, x_{i+1})$.

$p_{ij}$ - Probability of surviving from age $x_i$ to age $x_j$.

$\hat{p}_{ij}$ - Estimate of the probability of surviving from age $x_i$ to age $x_j$.

$p_{0i}$ - Probability of surviving from age $x_0$ to age $x_i$.

$p_{0j}$ - Estimate of the probability of surviving from age $x_0$ to age $x_j$.

$q_i$ - Probability of death in the interval $(x_i, x_{i+1})$.

$\hat{q}_i$ - Estimate of the probability of dying in age interval $(x_i, x_{i+1})$.

$S.E.$ - Standard error.

$S.E. (\text{diff.})$ - Standard error of a difference.

$S_{e_i}$ - Sample standard error of $\hat{e}_i$.

$S_{q_i}$ - Sample standard error of $\hat{q}_i$. 
Sample standard deviation of \( p_{01} \).

Sample variance.

Sample variance of \( e_\alpha \).

Sample variance of \( q_i \).

Sample variance of \( p_i \).

Sample variance of \( p_{01} \).

Sample variance of \( Y_\alpha \).

Sample variance of \( \bar{Y}_\alpha \).

Total number of years lived by the life table population beyond age \( x_i \).

Lower limit of the final age interval in a life table.

Lower limit of age interval \((x_{i},x_{i+1})\).

Upper limit of age interval \((x_{i},x_{i+1})\).

Length of life beyond age \( x_\alpha \) of the \( k \)-th individual in the group of \( \alpha \), for \( k=1,2,\ldots,\alpha \).

Mean length of life beyond age \( x_\alpha \).

Formulas

Estimation and hypothesis testing concerning probability \( q_i \):

\[
S^2_{q_i} = S^2_{p_i}
\]

\[
S^2_{q_i} = \frac{1}{D_i} \hat{q}_i^2 (1-\hat{q}_i)
\]

\[
\Pr\{\hat{q}_i - 1.96 S^2_{q_i} < q_i < \hat{q}_i + 1.96 S^2_{q_i}\} = .95
\]

\[
Z = \frac{\hat{q}_0(1960) - \hat{q}_0(1970)}{S.E.\{\hat{q}_0(1960) - \hat{q}_0(1970)\}}
\]
Hypothesis testing concerning survival probability $p_{ij}$:

$$p_{ij} = p_i p_{i+1} \cdots p_{j-1}$$  \hspace{1cm} (3.1)

$$p_{ij} = (1-q_i)(1-q_{i+1}) \cdots (1-q_{j-1}) .$$  \hspace{1cm} (3.2)

$$s_{p_{ij}}^2 = \frac{\hat{p}_{ij}}{p_{ij}} \sum_{h=i}^{j-1} \hat{p}_h \frac{s_{p_h}^2}{p_h}$$ \hspace{1cm} (3.7)

$$z = \frac{\hat{P}_{0.20}(U.S.) - \hat{P}_{0.20}(Cal.)}{S.E.(\text{diff.})} .$$  \hspace{1cm} (3.7a)

NOTE: For the cohort life table, $\hat{p}_{ij}$ is computed directly from

$$\hat{p}_{ij} = \frac{\hat{p}_j}{\hat{p}_i} ,$$  \hspace{1cm} (3.5)

with the variance given by:

$$s_{\hat{p}_{ij}}^2 = \frac{1}{\hat{p}_j} \hat{p}_{ij} (1-\hat{p}_{ij}) .$$ \hspace{1cm} (3.10)

Mean life time and expectation of life:

$$\bar{y}_\alpha = \frac{1}{\lambda} \sum_{k=1}^{\lambda} \frac{\lambda_k}{\lambda} \frac{1}{\lambda} \sum_{k=1}^{\lambda} Y_{ak}$$  \hspace{1cm} (4.1)

$$\bar{y} = \frac{L + L \alpha + \cdots + L \alpha^w}{\lambda}$$  \hspace{1cm} (4.13)

$$\bar{y}_\alpha = \frac{e^{\lambda \alpha}}{\alpha}$$  \hspace{1cm} (4.2)
Variance of the observed expectation of life:

\[ S_{\alpha}^2 = \frac{1}{k} \sum_{i=\alpha}^{w} [(x_i - x_{\alpha} + a_i n_i) - \hat{e}_i]^2 d_i . \quad (4.14) \]

\[ S_{e_i}^2 = S_{Y_i}^2 = \frac{1}{k} \sum_{i=1}^{W-1} S_{Y_i}^2, \quad (4.15) \]

\[ S_{e_i}^2 = \sum_{i=\alpha}^{W-1} \hat{e}_i^2 [(1-a_i)n_i + \hat{e}_{i+1}]^2 \frac{s_{\alpha}^2}{p_i} . \quad (4.27) \]
Crude probability - The probability of death from a specific cause in the presence of competition of all other risks acting in a population.

Net probability - The probability of death if a specific risk is the only risk in effect in a population, or conversely, the probability of death if a specific risk is eliminated from a population.

Partial crude probability - The probability of death from a specific cause when another risk is (or risks are) eliminated from a population.

Risk and cause - Both terms may refer to the same condition but are different on the time scale relative to the occurrence of death. Prior to death the condition in question is a risk; after death the condition is a cause (provided, of course, this is the condition from which an individual dies).

Cohort multiple decrement table - A cohort multiple decrement table records the mortality experience by cause of a well defined cohort of people from birth to the death of the last person of the group.

Current multiple decrement table - A current multiple decrement table is the one derived from the mortality experience by cause of a population of all ages during a current year.

Formulas

Age specific death rate:

\[ M_i = \frac{D_i}{(N_i - D_i)n_i + a_i n_i D_i} \]

(2.1)

\[ M_i = \frac{D_i}{P_i} \]

(2.3)

Estimate of the probability of dying:

\[ \hat{q}_i = \frac{D_i}{N_i} \]

(2.4)
Relationship between $M_1$ and $q_1$:

$$q_1 = \frac{n_1 M_1}{1 + (1-a_i) n_1 M_1}$$  \hspace{1cm} (2.5)

Age-cause-specific death rate:

$$M_{1\delta} = \frac{D_{1\delta}}{p_1}, \quad \delta = 1, \ldots, r.$$  \hspace{1cm} (2.7)

Estimate of the crude probability of dying from risk $R_0$:

$$\hat{Q}_{i\delta} = \frac{n_{1i\delta}}{1 + (1-a_i) n_{1i\delta}}.$$  \hspace{1cm} (2.9)

$$\hat{Q}_{i\delta} = \frac{D_{1\delta}}{D_i} q_i.$$  \hspace{1cm} (2.9a)

Relationship between the probability of dying $q_1$ and the crude probabilities, $Q_{i\delta}$:

$$\hat{Q}_{i1} + \ldots + \hat{Q}_{ir} = \hat{q}_i.$$  \hspace{1cm} (2.10)

Variance of the estimate of crude probability $\hat{Q}_{i\delta}$:

$$\text{Var}(\hat{Q}_{i\delta}) = \frac{1}{N_i} Q_{i\delta} (1 - Q_{i\delta}).$$  \hspace{1cm} (2.11)

$$\text{Var}(\hat{Q}_{i\delta}) = \frac{1}{N_i} \hat{Q}_{i\delta}(1 - \hat{Q}_{i\delta}) = \frac{1}{D_{1\delta}} \hat{Q}_{i\delta}^2(1 - \hat{Q}_{i\delta}).$$  \hspace{1cm} (2.12)
Standard deviation (standard error) of the estimate $\hat{Q}_{i6}$:

$$S.D. (\hat{Q}_{i6}) = \sqrt{\frac{1}{n_{i6}} \frac{\hat{Q}_{i6}^2}{1 - \hat{Q}_{i6}} (1 - \hat{Q}_{i6})}.$$  \hfill (2.13)

Standard deviation (standard error) of the estimate $q_i$:

$$S.D. (q_i) = \sqrt{\frac{1}{D_i} \frac{\hat{q}_i^2}{1 - \hat{q}_i}}.$$  \hfill (2.14)

Covariance between $\hat{Q}_{i1}$ and $\hat{Q}_{i2}$ from the same population:

$$Cov(\hat{Q}_{i1}, \hat{Q}_{i2}) = -\frac{1}{N_i} \hat{Q}_{i1} \hat{Q}_{i2} = -\frac{1}{p_{i1}^2} \hat{Q}_{i1}^2 \hat{Q}_{i2}.$$  \hfill (4.2)

Standard deviation (standard error) of the difference $\hat{Q}_{i1} - \hat{Q}_{i2}$:

$$S.D. (\hat{Q}_{i1} - \hat{Q}_{i2}) = \sqrt{S_{\hat{Q}_{i1}}^2 + S_{\hat{Q}_{i2}}^2 - 2 \text{Cov}(\hat{Q}_{i1}, \hat{Q}_{i2})}.$$  \hfill (4.4)

Critical ratio for comparing $\hat{Q}_{i1}$ between two populations:

$$z = \frac{\hat{Q}_{45,1}^A - \hat{Q}_{45,1}^S}{\sqrt{S_{\hat{Q}_{45,1}^A}^2 + S_{\hat{Q}_{45,1}^S}^2}}.$$  \hfill (4.1)
CHAPTER 8

Net probability:

- \( \hat{q}_{i,1} \): The probability of dying in \((x_i, x_{i+1})\) when \(R_i\) is eliminated as a risk of death.
- \( q_{i1} \): The probability of dying in \((x_i, x_{i+1})\) when \(R_i\) is the only risk acting in a population.
- \( \hat{q}_{i} - \hat{q}_{i,1} \): Reduction in the probability of dying in interval \((x_i, x_{i+1})\) due to the presence of risk \(R_i\).

Expectation of life:

- \( \hat{e}_{i,1} \): The expectation of life at age \(x_i\) when \(R_i\) is eliminated as a risk of death.
- \( \hat{e}_{i,1} - \hat{e}_i \): Reduction in the expectation of life at age \(x_i\) due to the presence of risk \(R_i\).

Formulas

The net probabilities:

\[
q_{i,1} = (q_i - q_{i1})(1 + \frac{1}{2} q_{i1}) \tag{2.1}
\]

\[
q_{i1} = q_{i1} \left(1 + \frac{1}{2} (q_i - q_{i1})\right) \tag{5.1}
\]

Computation of the estimate \( \hat{q}_{i,1} \):

\[
M_{i1} = \frac{D_{i1}}{p_i} \tag{2.3}
\]

\[
\hat{q}_i = \frac{n_i M_i}{1 + (1-M_i) n_i M_i} \tag{2.4}
\]

\[
\hat{Q}_{i1} = \frac{n_i M_{i1}}{1 + (1-M_i) n_i M_{i1}} \tag{2.5}
\]

\[
\hat{q}_{i,1} = (\hat{q}_i - \hat{Q}_{i1})(1 + \frac{1}{2} \hat{Q}_{i1}) \tag{2.6}
\]
Formulas used in the construction of life tables when \( R_1 \) is eliminated:

\[
d_{0.1} = l_{0.1} q_{0.1} \quad (3.1)
\]

\[
l_{1.1} = l_{0.1} - d_{0.1} \quad (3.2)
\]

\[
L_{0.1} = (l_{0.1} - d_{0.1}) + a_0 d_{0.1} \quad (3.3)
\]

\[
\hat{e}_{95.1} = \frac{p_{95}}{D_{95} - D_{95.1}} \quad (3.4)
\]

\[
T_{95.1} = l_{95.1} \hat{e}_{95.1} \quad (3.5)
\]

\[
L_{95.1} = T_{95.1} \quad (3.6)
\]

\[
d_{95.1} = l_{95.1} \quad (3.7)
\]

\[
\hat{q}_{95.1} = 1.00000 \quad (3.8)
\]

\[
T_{i.1} = L_{i.1} + \ldots + L_{95.1} \quad (3.9)
\]

\[
\hat{e}_{i.1} = \frac{T_{i.1}}{\bar{k}_{i.1}} \quad (3.11)
\]
CHAPTER 9

- The number of years since admission to a follow-up study.

- The number of patients alive at the beginning of the interval \((x, x+1)\), \(N_x = m_x + n_x\).

- The number of patients who entered a follow-up study more than \(x+1\) years before the closing date who will be observed for the entire interval \((x, x+1)\).

- The number of patients who entered the study less than \(x+1\) years before the closing date and are due to withdraw in the interval \((x, x+1)\).

- The number of patients among \(m_x\) dying in the interval \((x, x+1)\).

- The number of patients among \(m_x\) surviving to the end of the interval \((x, x+1)\).

- The number of patients among \(n_x\) who will die before the time of withdrawal.

- The number of patients among \(w_x\) who survive to the time of withdrawal.

- Estimate of the probability of surviving from admission to the interval \((0, x)\).

- Estimate of the expectation of life at \(x=\alpha\).

- Total force of mortality at time \(t\).

- The force of mortality for risk \(R_\delta\) at time \(t\).

- The number of patients among \(d_x\) dying from risk \(R_\delta\).

- The number among \(d'_x\) dying from risk \(R_\delta\).

Formulas

Binomial distribution among those not due for withdrawal in \((x, x+1)\):

\[
E(s_x | m_x) = m_x p_x \quad \text{and} \quad E(d_x | m_x) = m_x (1-p_x)
\]
Binomial distribution among those due for withdrawal in \((x, x+1)\):

\[
c_2 \frac{\nu_x}{x} \frac{1}{(1-p_x)^x} \frac{d_x}{x} \]

(2.5)

\[
E(n_x) = n_x \frac{\nu_x}{x} \quad \text{and} \quad E(d_x' | n_x) = n_x (1-p_x)^x ,
\]

(2.6)

Estimate of the probability of survival \(p_x\) and its sample variance:

\[
\hat{p}_x = \left[ \frac{-\frac{1}{2}d_x' + \sqrt{\frac{1}{2}d_x'^2 + 4(N_x - \nu_{n_x})(s_x + \nu_{n_x})}}{2(N_x - \nu_{n_x})} \right]^2
\]

(2.8)

\[
\hat{q}_x = 1 - \hat{p}_x , \quad x = 0, 1, \ldots, y-1.
\]

(2.9)

\[
\hat{s}^2 = \frac{\hat{p}_x \hat{q}_x}{m_x}
\]

(2.10)

where

\[
m_x = n_x + n_x (1-p_x)^{-1}
\]

(2.11)
Estimate of survival probability $p_{0x}$ and its sample variance:

$$
\hat{p}_{0x} = \hat{p}_0 \hat{p}_1 \cdots \hat{p}_{x-1}, \quad x = 1, 2, \ldots, y. \tag{2.12}
$$

$$
S_{p_{0x}}^2 = \hat{p}_{0x}^2 \sum_{u=0}^{x-1} \hat{p}_u S_{p_u}^2. \tag{2.13}
$$

Estimate of expectation of life and its sample variance:

$$
\hat{e}_a = \frac{1}{2} + \hat{p}_a + \hat{p}_a \hat{p}_{a+1} + \cdots + \hat{p}_a \hat{p}_{a+1} \cdots \hat{p}_{y-1} + \hat{p}_y \left( \frac{\hat{p}_t}{1 - \hat{p}_t} \right), \tag{2.19}
$$

$$
S_{\hat{e}_a}^2 = \sum_{x=a}^{y-1} \hat{p}_a x \left( \hat{e}_{x+1} + \frac{1}{2} \right)^2 S_{p_x}^2 + \hat{p}_a \left( \hat{e}_{t+1} + \frac{1}{2} + \frac{\hat{p}_y}{(1 - \hat{p}_t)^2} \right)^2 S_{p_t}^2, \quad a \leq t. \tag{2.23}
$$

$$
S_{\hat{e}_a}^2 = \sum_{x=a}^{y-1} \hat{p}_a x \left( \hat{e}_{x+1} + \frac{1}{2} \right)^2 S_{p_x}^2 + \frac{\hat{p}_y^2}{(1 - \hat{p}_t)^4} S_{p_t}^2, \quad a > t. \tag{2.24}
$$

Forces of mortality:

$$
\mu(\tau; 1) + \cdots + \mu(\tau; \tau) = \mu(\tau) \tag{3.1}
$$
Crude probability of dying:

\[
Q_{x \delta}(t) = \frac{\mu_{x,\delta}}{\mu(x)} \left[ 1 - p_x(t) \right], \quad 0 < t \leq 1; \quad \delta = 1, \ldots, r. \tag{3.2}
\]

\[
Q_{x1}(t) + \cdots + Q_{x \delta}(t) + p_x(t) = 1, \quad 0 < t \leq 1. \tag{3.3}
\]

\[
Q_{x \delta}(t) = \frac{\mu_{x,\delta}}{\mu(x)} \left[ 1 - p_x^{1/2} \right] = Q_{x \delta} \left[ 1 + p_x^{1/2} \right]^{-1}, \quad \delta = 1, \ldots, r. \tag{3.4}
\]

Net probabilities of dying:

\[
q_{x \delta} = Q_{x \delta} \left[ 1 + \frac{1}{2} (\eta_x - Q_{x \delta}) + \frac{1}{6} (\eta_x - Q_{x \delta}) (2\eta_x - Q_{x \delta}) \right]; \tag{3.6}
\]

\[
q_{x \delta} = (Q_x - Q_{x \delta}) \left[ 1 + \frac{1}{2} Q_{x \delta} + \frac{1}{6} Q_{x \delta} (Q_x + Q_{x \delta}) \right], \quad \delta = 1, \ldots, r. \tag{3.7}
\]

Partial crude probability of dying:

\[
Q_{x \delta.1} = Q_{x \delta.1} + \frac{1}{2} Q_{x \delta} + \frac{1}{6} Q_{x1} (Q_x + Q_{x1}) \tag{3.8}
\]
Multinomial distribution among those not due for withdrawal in \((x, x+1)\):

\[
C_1 p_x^s Q_{x1}^d \ldots Q_{xr}^d ,
\]

(3.10)

\[
E(s_x | m_x) = m_x p_x \quad \text{and} \quad E(d_x | m_x) = m_x Q_x^d
\]

(3.11)

Multinomial distribution among those due for withdrawal in \((x, x+1)\):

\[
C_2 p_x^{1 + r} \prod_{\delta=1}^r Q_x^d (1 + p_x)^{-1}
\]

(3.13)

\[
E(w_x | n_x) = n_x p_x^{1 + r} \quad \text{and} \quad E(d_x^r | n_x) = n_x Q_x^d (1 + p_x)^{-1}
\]

(3.14)

Estimates of probabilities of dying:

\[
\hat{q}_{x\delta} = \frac{\hat{D}_{x\delta}}{\hat{D}_x}, \quad \delta = 1, 2, \ldots, r, \quad x = 0, 1, \ldots, y - 1.
\]

(3.18)

\[
\hat{q}_{x\delta} = \hat{q}_{x\delta} [1 + \frac{1}{6} (\hat{q}_{x\delta} - \hat{q}_{x\delta})] + \frac{1}{6} [q_{x\delta} - \hat{q}_{x\delta}] (2q_{x\delta} - \hat{q}_{x\delta})
\]

(3.19)

\[
\hat{\tau}_{x\delta} = (\hat{q}_{x\delta} - \hat{q}_{x\delta}) [1 + \frac{1}{6} \hat{\tau}_{x\delta} + \frac{1}{6} \hat{\tau}_{x\delta} (q_{x\delta} + \hat{q}_{x\delta})] , \quad \delta = 1, \ldots, r,
\]

(3.20)

and

\[
\hat{\tau}_{x\delta \cdot 1} = \hat{q}_{x\delta} [1 + \frac{1}{6} \hat{\tau}_{x\delta} + \frac{1}{6} \hat{\tau}_{x\delta} (q_{x\delta} + \hat{q}_{x\delta})] , \quad \delta = 2, \ldots, r;
\]

\[
x = 0, 1, \ldots, y - 1.
\]

(3.21)
Lost cases:

\[ u(x; r) + o(\Delta) = \Pr\{ \text{a patient will be lost to the study in } (\tau, \tau + \Delta) \text{ due to follow-up failure} \}, \]
\[ x < \tau < x + 1. \]  

\[ p_x = \Pr\{ \text{a patient alive at time } x \text{ will remain alive and under observation at time } x + 1 \}. \]  

\[ q_x = 1 - p_x = \Pr\{ \text{a patient alive at time } x \text{ will either die or be lost to the study due to follow-up failure in interval } (x, x + 1) \}. \]  

\[ q_{x,r} = \Pr\{ \text{a patient alive at time } x \text{ will be lost to the study in } (x, x + 1) \}. \]  

\[ q_{x,r} = \Pr\{ \text{a patient alive at time } x \text{ will die in interval } (x, x + 1) \text{ if the risk } R_r \text{ of being lost is eliminated} \}. \]  

\[ 1 - q_{x,r} = \Pr\{ \text{a patient alive at x will survive to time } x + 1 \text{ if the risk } R_r \text{ of being lost is eliminated} \}. \]  

\[ q_{x,r} = \Pr\{ \text{a patient alive at } x \text{ will die in } (x, x + 1) \text{ from risk } R_6 \text{ if the risk } R_r \text{ of being lost is eliminated} \}. \]
APPENDIX I

\( a_i \) - The fraction of last age interval of life. The expected fraction of the interval \( (x_i, x_i + n_i) \) lived by an individual who dies at an age included in the interval.

\( \tau_i \) - The fraction of the interval \( (x_i, x_i + n_i) \) lived by an individual who dies at an age included in the interval. \( \tau_i \) is a random variable whose expectation is \( a_i \), or \( E(\tau_i) = a_i \).

\( M_i \) - Age specific death rate. The ratio of the observed number of deaths (\( D_i \)) to the total number of years lived in the interval \( (x_i, x_i + n_i) \) by those who are alive at \( x_i \). \( M_i \) is a random variable.

\( m_i \) - (Theoretical) age specific death rate. The ratio of the expected number of deaths to the expected number of years lived in the interval \( (x_i, x_i + n_i) \) by those who are alive at \( x_i \). \( m_i \) is an unknown theoretical value.

\( q_i \) - Probability of dying in interval \( (x_i, x_i + n_i) \).

\( \mu(x) \) - Force of mortality (mortality intensity function) age \( x \).

Formulas

Relationship between \( q_i \) and \( m_i \):

\[ q_i = 1 - \exp\left\{-\int_0^{n_i} \mu(x_i + \xi) \, d\xi \right\} \]  \hspace{1cm} (1)

\[ m_i = \frac{n_i}{\int_0^{n_i} \exp\left\{-\int_0^y \mu(x_i + \xi) \, d\xi \right\} \, dy} \]  \hspace{1cm} (2)

\[ g(t)dt = \frac{\int_0^{n_i} \mu(x_i + \xi) \, d\xi \, t \, n_i \, dt}{q_i} \]  \hspace{1cm} (4)

\[ 0 \leq t \leq 1 \]
\[ a_i = E(t_i) = \int_0^1 t g(t) dt \] (6)

\[ q_i = \frac{n_i m_i}{1 + (1 - a_i) n_i m_i} \] (9)

\[ q_x = \frac{m_x}{1 + (1 - a'_x) m_x} \] (10)
APPENDIX II

\( Y_\alpha \) - The future life time beyond age \( x_\alpha \). This is a random variable.  

\( e_\alpha \) - The true expectation of life beyond age \( x_\alpha \). \( Y_\alpha \) is a random variable whose expectation is \( e_\alpha \), or \( E(Y_\alpha) = e_\alpha \).  

\( f(y_\alpha) \) - Probability density function of \( Y_\alpha \). The product \( f(y_\alpha)dy_\alpha \) is the probability that an individual alive at \( x_\alpha \) will survive the period \( (x_\alpha, x_\alpha + y_\alpha) \) and then die in the interval \( (x_\alpha + y_\alpha, x_\alpha + y_\alpha + dy_\alpha) \).  

\( X \) - The life span of an individual. It is a continuous random variable.  

\( F_X(x) \) - Distribution function of the length of life \( X \). It is the probability of dying prior to, or at, age \( x \).  

\( g_x \) - The number of individuals surviving to age \( x \).  

\( p_{ij} \) - Probability that an individual alive at age \( x_i \) will survive to age \( x_j \), for \( i < j \).  

\( 1-p_{ij} \) - Probability that an individual alive at age \( x_i \) will die before age \( x_j \).  

\( p_{0x} \) - Probability that one individual alive at age \( 0 \) will survive to age \( x \).  

\( p_{01}q_i \) - Probability that an individual alive at age \( 0 \) will die in the interval \( (x_i, x_{i+1}) \), \( i=0,1,\ldots,w \).  

\( p_{ai}q_i \) - Probability that an individual alive at age \( x_a \) will die in the interval \( (x_i, x_{i+1}) \) subsequent to \( x_a \).  

\( \rho_{x_i, x_j} \) - Correlation between \( x_i \) and \( x_j \).  

\( \sigma_{x_i, x_j} \) - Covariance between \( x_i \) and \( x_j \).  

Formulas

Distribution of the length of life \( X \) and the number of survivors \( x \):

\[ \mu(x)\Delta + o(\Delta) = \Pr\{ \text{an individual alive at age } x \text{ will die in interval } (x, x+\Delta) \} \]  

(2.1)

\[ F_X(x) = \Pr\{ X < x \} \]  

(2.2)
\[
1 - F_X(x) = e^{-\int_0^x u(t)\,dt} = P_{0x}
\]
\[
Pr(l_x = k) = \frac{\lambda_0^k}{k!(\lambda_0 - k)!} \frac{k^k}{P_{0x}(1-P_{0x})^{\lambda_0-k}}, \quad k=0,1,\ldots,\lambda_0
\]
\[
E(l_x \mid \lambda_0) = \lambda_0 P_{0x}
\]
\[
\sigma^2_{l_x \mid \lambda_0} = \lambda_0 P_{0x}(1-P_{0x})
\]
\[
P_{ij} = \exp\left\{-\int_{x_i}^{x_j} u(t)\,dt\right\}, \quad \text{for } i<j
\]
\[
f_X(x) = \frac{dF_X(x)}{dx} = u(x)e^{\int_0^x u(t)\,dt}
\]
\[
= 0 \quad \text{for } x<0
\]

Gompertz distribution:

\[
u(t) = Be^t
\]
\[
f(x) = Be^x e^{-B[e^x-1]/\ln c}
\]
\[
F_X(x) = 1 - \exp\left(-\frac{B}{\ln c} (e^x-1)\right)
\]
Makeham distribution:

\[ \mu(t) = A + B e^{ct} \]  \hspace{1cm} (2.26)

\[ f(x) = [A + B e^{x}] \exp\left\{-\left[ A + B (e^{x} - 1)/\ln e \right] \right\} \]  \hspace{1cm} (2.27)

\[ F_X(x) = 1 - \exp\left\{-\left[ A + B (e^{x} - 1)/\ln e \right] \right\} \]  \hspace{1cm} (2.28)

Weibull distribution:

\[ \mu(t) = \mu a e^{a - 1} \]  \hspace{1cm} (2.29)

\[ f(x) = \mu a x^{a-1} e^{-\mu x} \]  \hspace{1cm} (2.29a)

\[ F_X(x) = 1 - e^{-\mu x} \]  \hspace{1cm} (2.30)

Exponential distribution:

\[ \mu(t) = \mu \text{ is a constant} \]

\[ f(x) = \mu e^{-\mu x} \]  \hspace{1cm} (2.31)

\[ F_X(x) = 1 - e^{-\mu x} \]  \hspace{1cm} (2.32)
Joint distribution of $\ell_1, \ell_2, \ldots, \ell_u$, and their correlation:

\[
\Pr(\ell_1 = k_1, \ell_2 = k_2, \ldots, \ell_u = k_u | \ell_0) = \frac{k_1!}{(k_1 + 1)!(k_u - k_i + 1)!} \frac{k_{i+1}^{k_i-k_{i+1}}}{p_i (1-p_i)^{k_i-k_{i+1}}} \]

\[
k_{i+1} = 0, 1, \ldots, k_i, \quad \text{with } k_0 = \ell_0 \tag{3.3}
\]

\[
\varepsilon_{\ell_i, \ell_j} = \frac{p_{o_1}(1-p_{o_1})}{p_{o_1}(1-p_{o_1})p_{o_j}(1-p_{o_j})} = \sqrt{\frac{p_{o_1}(1-p_{o_1})}{p_{o_1}(1-p_{o_1})}} \tag{3.8}
\]

\[
p_0 \varepsilon_0^2 + \cdots + p_u \varepsilon_u^2 = 1. \tag{4.2}
\]

Joint distribution of $d_0, d_1, \ldots, d_u$, and their correlation:

\[
\Pr(d_0 = \delta_0, \ldots, d_u = \delta_u) = \frac{\delta_0!}{\delta_0! \cdots \delta_u!} (p_0 \varepsilon_0^2)^{\delta_0} \cdots (p_u \varepsilon_u^2)^{\delta_u} \tag{4.3}
\]

\[
E(d_1 | \ell_0) = \varepsilon_0 p_{o_1} q_{i_1} \tag{4.4}
\]

\[
\sigma_{d_1}^2 = \varepsilon_0 p_{o_1} q_{i_1} (1-p_{o_1} q_{i_1}) \tag{4.5}
\]

\[
\sigma_{d_i, d_j} = -\varepsilon_0 p_{o_1} q_{i_1} p_{o_j} q_{j} \quad \text{for } i \neq j; i, j = 0, 1, \ldots, u. \tag{4.6}
\]

Maximum likelihood estimates of $p_0, p_1, \ldots, p_u$:

\[
L = \prod_{i=0}^{u-1} \frac{\ell_i!}{\ell_{i+1}!(\ell_i - \ell_{i+1})!} p_i^{\ell_i+1} (1-p_i)^{\ell_i-\ell_{i+1}} \tag{5.2}
\]

\[
\hat{p}_j = \frac{\ell_j+1}{\ell_j} \quad j = 0, 1, \ldots, u-1 \tag{5.5}
\]

\[
E[\hat{p}_j] = E[\frac{\ell_j+1}{\ell_j}] = E[\frac{1}{\ell_j} E(\ell_j+1 | \ell_j)] = \hat{p}_j \tag{5.6}
\]
\[ \sigma^2_{p_j} = \left( \frac{1}{\bar{x}_j} \right) p_j (1-p_j) = \sigma^2_{\hat{a}_j} \]  \hspace{1cm} (5.8)

When \( \bar{x}_0 \) is large, \[ \sigma^2_{p_j} = \frac{1}{E(\bar{x}_j)} p_j (1-p_j) \]  \hspace{1cm} (5.9)

\[ \sigma_{p_j, p_k} = 0 \]  \hspace{1cm} (5.11)

\[ \sigma_{p_{a_j}, p_{a_k}} = E \left( \frac{1}{\bar{x}_a} \right) p_{a_k} (1-p_{a_j}) \quad \alpha < j \leq k \]  \hspace{1cm} (5.12)

**Expectation of life and its estimate:**

\[ e_\alpha = \int_0^\infty y f(y_\alpha) \, dy_\alpha = \int_0^\infty y_\alpha e^{-\mu(x_\alpha + y_\alpha)} \mu(x_\alpha + y_\alpha) \, dy_\alpha \]  \hspace{1cm} (6.6)

\[ \sigma^2_{e_\alpha} = \int_0^\infty (y_\alpha - e_\alpha)^2 f(y_\alpha) \, dy_\alpha \]  \hspace{1cm} (6.7)

\[ \bar{y}_\alpha = \hat{e}_\alpha \]  \hspace{1cm} (6.13)

\[ \hat{e}_\alpha = a_{\alpha} n_\alpha + \sum_{i=\alpha+1}^w c_i \frac{b_i}{k_\alpha} = a_{\alpha} n_\alpha + \sum_{i=\alpha+1}^w c_i \hat{p}_{a_i} \]  \hspace{1cm} (6.15)

\[ e_\alpha = a_{\alpha} n_\alpha + \sum_{i=\alpha+1}^w c_i p_{a_i} , \quad \alpha=0,1,\ldots,w. \]  \hspace{1cm} (6.16)

\[ \sigma^2_{e_\alpha} = \sum_{i=\alpha}^{w-1} p_{a_i}^2 (e_{i+1} + (1-a_{i+1})n_{i+1})^2 \sigma^2_{q_i} \quad \alpha=0,1,\ldots,w-1. \]  \hspace{1cm} (6.21)
APPENDIX III

$Q_{i\delta}$ - Crude probability of dying from risk $R_\delta$.

$Q_{i\delta} = \Pr\{\text{an individual alive at age } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from } R_\delta \text{ in the presence of other competing risks operating in the population}\}$

$q_{i\delta}$ - Net probability of dying from risk $R_\delta$.

$q_{i\delta} = \Pr\{\text{an individual alive at age } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ when } R_\delta \text{ is the only risk operating in the population}\}$

$q_{i,\delta}$ - Net probability of dying when $R_\delta$ is eliminated as a risk of death.

$q_{i,\delta} = \Pr\{\text{an individual alive at age } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ when } R_\delta \text{ is eliminated as a risk of death}\}$

$q_{i\delta,1}$ - Partial crude probability of dying

$q_{i\delta,1} = \Pr\{\text{an individual alive at age } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from } R_\delta \text{ when } R_1 \text{ is eliminated as a risk of death}\}$

$q_{i\delta,12}$ - Partial crude probability of dying

$q_{i\delta,12} = \Pr\{\text{an individual alive at age } x_i \text{ will die in the interval } (x_i, x_{i+1}) \text{ from } R_\delta \text{ when } R_1 \text{ and } R_2 \text{ are eliminated as risks of death}\}$

$p_i$ - Probability of surviving the interval $(x_i, x_{i+1})$

$q_i$ - Probability of dying in the interval $(x_i, x_{i+1})$

$R_{\delta\epsilon}$ - Interaction between risks $R_\delta$ and $R_\epsilon$

$\mu(t;\delta)$ - Force of mortality associated with risk $R_\delta, \delta = 1, \ldots, r$

$\mu(t)$ - Total force of mortality.

$\mu(t) = \mu(t;1) + \ldots + \mu(t;r)$

$\mu(t;\delta,\epsilon)$ - Force of mortality associated with the interaction $R_{\delta\epsilon}$
Relationship between three types of probabilities:

\[ u(t;\delta) + o(\Delta) = \Pr\{\text{an individual alive at time } t \text{ will die in interval } (t, t+\Delta) \text{ from risk } R_\delta \}, \delta=1,\ldots,r \]

\[ u(t;1) + \cdots + u(t;r) = u(t) \]  

Proportionality Assumption: \[ \frac{u(t;\delta)}{u(t)} = c_\delta \]  

\[ q_{i\delta} + \cdots + q_{ir} = q_i \quad i=0,1,\ldots. \]  

\[ q_{i\delta} = 1 - p_i \frac{Q_{1\delta}}{q_i} \quad \delta=1,\ldots,r. \]  

When \( q_i \) is extremely small, \( q_{i\delta} = Q_{i\delta} [1 + \frac{1}{2} (q_i-Q_{i\delta})] \)

\[ q_{i\cdot\delta} = 1 - p_i \frac{(q_i-Q_{i\delta})}{q_i} \]  

\[ q_{i\cdot\delta} = (q_i-Q_{i\delta}) [1 + \frac{1}{2} Q_{i\delta} + \frac{1}{6} Q_{i\delta} (q_i+Q_{i\delta})] \]  

\[ Q_{i\delta\cdot1} = Q_{i\delta} [1 + \frac{1}{2} Q_{i1} + \frac{1}{6} Q_{i1} (q_i+Q_{i1})] \quad \delta=2,\ldots,r. \]  

\[ u(t;\delta,\epsilon) + o(\Delta) = \Pr\{\text{an individual alive at time } t \text{ will die in interval } (t, t+\Delta) \text{ from } R_{\delta\epsilon} \} \]

\[ \sum_{\delta=1}^{r} \frac{u(t;\delta)}{\sum_{\epsilon=\delta+1}^{r} u(t;\delta,\epsilon)} = u(t) \]  

\[ q_{i\cdot1} = 1 - p_i \frac{(q_i-Q_{i1}-\frac{1}{2} Q_{i1\epsilon})}{q_i} \]
\[ q_{4.1} = (q_{4} - Q_{11} - \sum_{\epsilon=2}^{r} Q_{41\epsilon} \epsilon) \left[ 1 + \frac{1}{2} (Q_{411} + \sum_{\epsilon=2}^{r} Q_{11\epsilon}) + \frac{1}{6} (Q_{411} + \sum_{\epsilon=2}^{r} Q_{11\epsilon}) (q_{4} + Q_{11} + \sum_{\epsilon=2}^{r} Q_{41\epsilon}) \right] \]  

(3.12)

\[ Q_{16.1} = \frac{Q_{16}}{q_{4} - Q_{11} - \sum_{\epsilon=2}^{r} Q_{41\epsilon}} q_{4.1} \]  

(3.16)
APPENDIX IV

\( d_{i6} \) - Number of deaths from cause \( R_6 \) in the interval \( (x_i, x_{i+1}) \).  
\[
\rho_{d_{i6}, d_{i6}} \mid \xi_i \] - Correlation coefficient between \( d_{i6} d_{i6} \), given \( \xi_i \).  
\[
\rho_{d_{i6}, d_{i6}} \] - Correlation coefficient between \( d_{i6} \) and \( d_{i6} \).  
\( s_{Q_{i6}} \) - Standard error of \( Q_{i6} \).  

Formulas

\[
d_{i1} + \cdots + d_{ir} = d_i \] (1.1)
\[
\xi_i = d_{i1} + \cdots + d_{ir} + \xi_{i+1} \] (1.3)
\[
Q_{i1} + \cdots + Q_{ir} = q_i \] (1.4)
\[
1 = Q_{i1} + \cdots + Q_{ir} + p_i \] (1.6)

Joint probability distribution of \( d_{i1}, \ldots, d_{ir}, \xi_{i+1} \) given \( \xi_i \):
\[
\frac{\xi_i!}{d_{i1}! \cdots d_{ir}! \xi_{i+1}!} Q_{i1} \cdots Q_{ir} p_i^{\xi_i+1} \] (1.7)

where \( d_{i1} + \cdots + d_{ir} + \xi_{i+1} = \xi_i \).

\[
E(d_{i6} \mid \xi_i) = \xi_i Q_{i6} \] (1.8)
\[
\text{Var}(d_{i6} \mid \xi_i) = \xi_i Q_{i6} (1 - Q_{i6}), \xi_i = 1, \cdots, r. \] (1.9)
Joint probability distribution of all the random variables $d_{i1}, \ldots, d_{ir}, \ell_{i+1}$. for $i=0, 1, \ldots, u$, given $\ell_0$:

\begin{equation}
\rho_{d_{i\delta}, d_{i\epsilon} | \ell_1} = \frac{\sqrt{Q_{i\delta} p_i}}{\sqrt{1 - Q_{i\delta}} (1 - Q_{i\epsilon})} (1.11)
\end{equation}

\begin{equation}
\rho_{d_{i\delta}, \ell_{i+1} | \ell_1} = \frac{\sqrt{Q_{i\delta} p_i}}{\sqrt{1 - Q_{i\delta}} (1 - p_i)} (1.13)
\end{equation}

\begin{equation}
\text{Var}(\ell_{i+1} | \ell_1) = \ell_1 p_1 q_1 (1.15)
\end{equation}

\begin{equation}
\prod_{i=0}^{u} \frac{\ell_1!}{d_{i1}! \cdots d_{ir}! \ell_{i+1}!} Q_{i1} d_{i1} \cdots Q_{ir} d_{ir} \ell_{i+1}^{\ell_{i+1}} (2.1)
\end{equation}

\begin{equation}
E(d_{i\delta}) = E(\ell_1) Q_{i\delta} = \ell_0 p_{0i} Q_{i\delta} (2.2)
\end{equation}

\begin{equation}
\text{Var}(d_{i\delta}) = \ell_0 p_{0i} Q_{i\delta} (1 - p_{0i} Q_{i\delta}), \quad i=0, \ldots, u. (2.6)
\end{equation}

\begin{equation}
\text{Cov}(d_{i\delta}, d_{i\epsilon}) = -\ell_0 p_{0i} Q_{i\delta} p_{0i} Q_{i\epsilon}, \quad \delta \neq \epsilon; \quad \delta, \epsilon = 1, \ldots, r; \quad i=0, \ldots, u. (2.7)
\end{equation}

\begin{equation}
\rho_{d_{i\delta}, d_{i\epsilon}} = -p_{0i} \sqrt{Q_{i\delta}} \sqrt{Q_{i\epsilon}} \sqrt{1 - p_{0i} Q_{i\delta}} \sqrt{1 - p_{0i} Q_{i\epsilon}} (2.9)
\end{equation}

\begin{equation}
\rho_{d_{i\delta}, d_{j\epsilon}} = -p_{0i} \sqrt{p_{ij}} \sqrt{Q_{i\delta}} \sqrt{Q_{j\epsilon}} \sqrt{1 - p_{0i} Q_{i\delta}} \sqrt{1 - p_{0j} Q_{j\epsilon}}, \text{ for } i \neq j (2.12)
\end{equation}
\begin{align}
\text{Cov}(d_{i\delta}, l_{j}) &= -\xi_{0}p_{O_{i}}q_{i\delta}p_{O_{j}} & (2.13) \\
\text{Cov}(l_{i}, d_{j\delta}) &= \xi_{0}(1-p_{O_{i}})p_{O_{j}}q_{i\delta} & \delta = 1, \ldots, r; \quad i < j; \ i, j = 0, 1, \ldots, \\
\sigma_{l_{i}, l_{j}} &= \xi_{0}(1-p_{O_{i}})p_{O_{j}} & i < j, \\
\text{Maximum likelihood estimates of } p_{i}, q_{i\delta}:
L &= \prod_{i=0}^{u} \frac{l_{i}!}{d_{i1}! \ldots d_{i\delta}! \ldots d_{i\delta+1}!} q_{i1} \ldots q_{i\delta} r_{i} p_{i} & (3.1) \\
\hat{q}_{i\delta} &= \frac{d_{i\delta}}{l_{i}}, \quad \delta = 1, \ldots, r \\
\hat{p}_{i} &= \frac{l_{i+1}}{l_{i}}, \quad i = 0, \ldots, u \\
\text{Var}(\hat{q}_{i\delta}) &= E\left(\frac{1}{l_{i}}\right) q_{i\delta}(1-q_{i\delta}) & \delta = 1, \ldots, r, \quad i = 0, \ldots, u \\
\text{Var}(\hat{p}_{i}) &= \text{Var}(\hat{q}_{i}) = E\left(\frac{1}{l_{i}}\right) p_{i} q_{i} & (3.16) \\
\text{Cov}(\hat{q}_{i\delta}, \hat{q}_{i\epsilon}) &= -E\left(\frac{1}{l_{i}}\right) q_{i\delta} q_{i\epsilon} & (3.17) \\
\text{Cov}(\hat{q}_{i\delta}, \hat{p}_{i}) &= -E\left(\frac{1}{l_{i}}\right) p_{i} q_{i\delta} & (3.18) \\
\hat{s}_{q_{i\delta}} &= \sqrt{\frac{1}{l_{i}} \hat{q}_{i\delta}(1-\hat{q}_{i\delta})} & \delta = 1, \ldots, r \\
\hat{s}_{q_{i}} &= \sqrt{\frac{1}{l_{i}} \hat{q}_{i}(1-\hat{q}_{i})} & \delta = 0, \ldots, u.
\end{align}